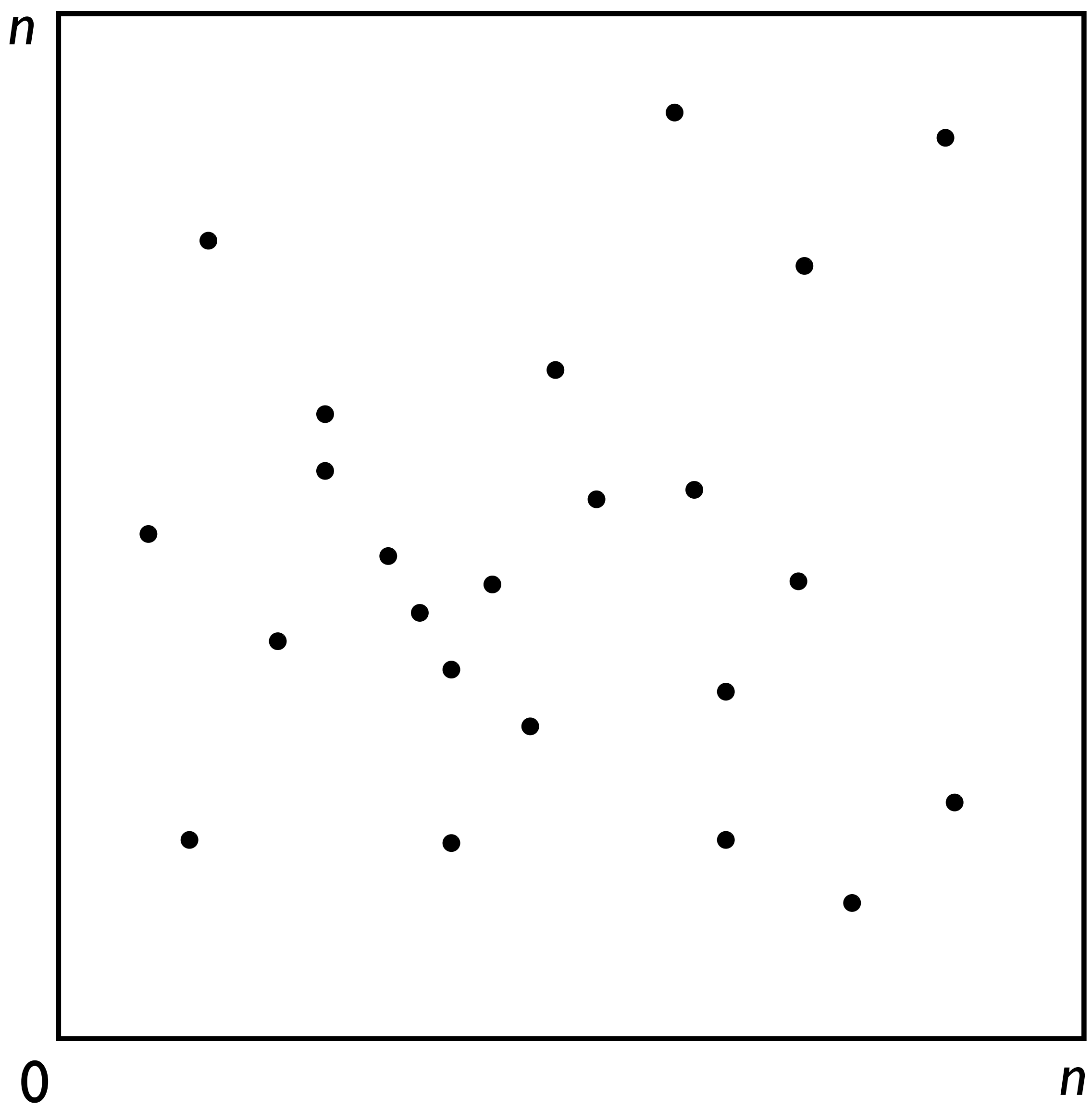
Space Complexity of 2-Dimensional Approximate Range Counting

Zhewei Wei and Ke Yi

Problem and Results

Problem Definition

- *P*: *n* points on a $n \times n$ grid and ε: error parameter.



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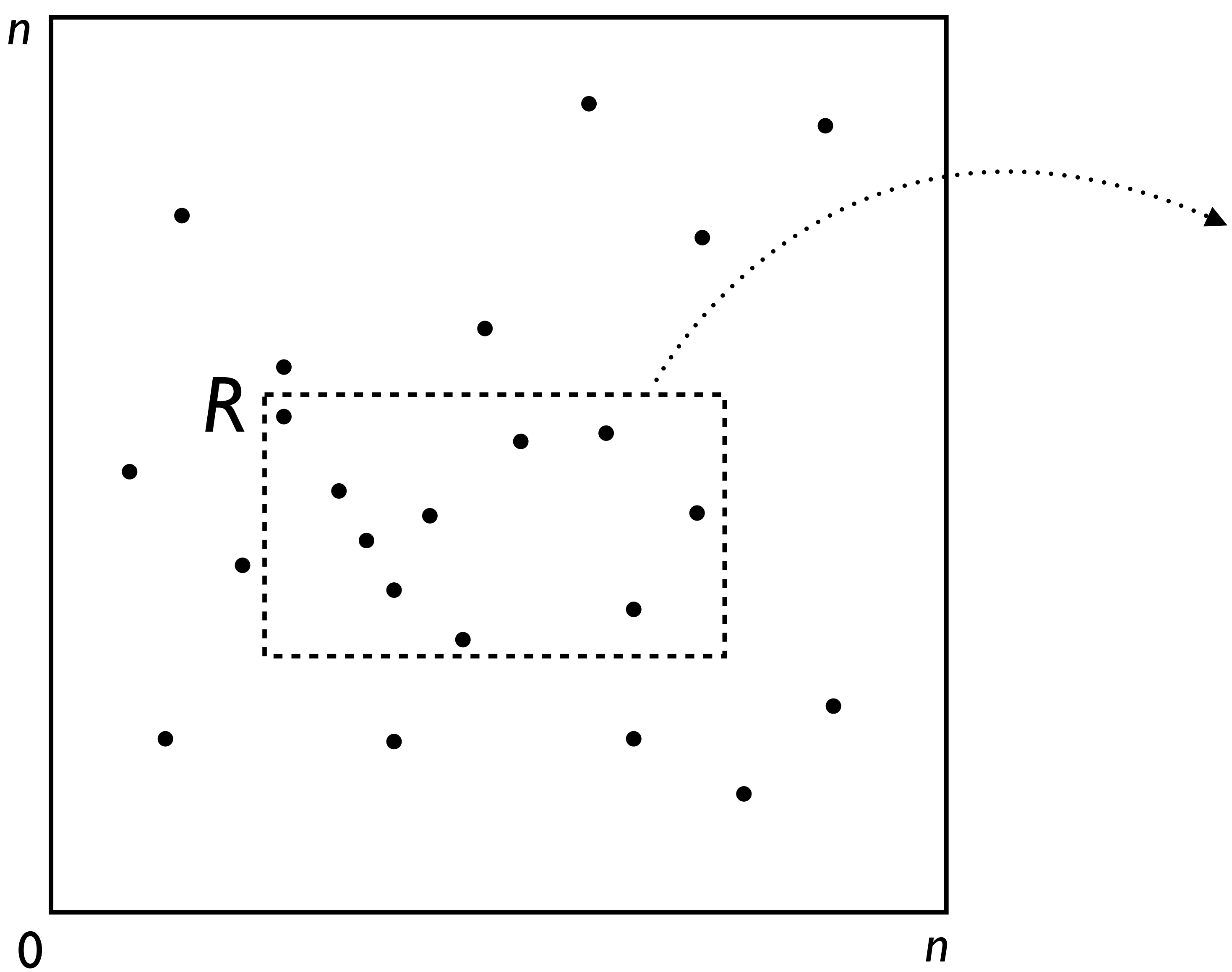

$|P \cap R| \pm \varepsilon n$

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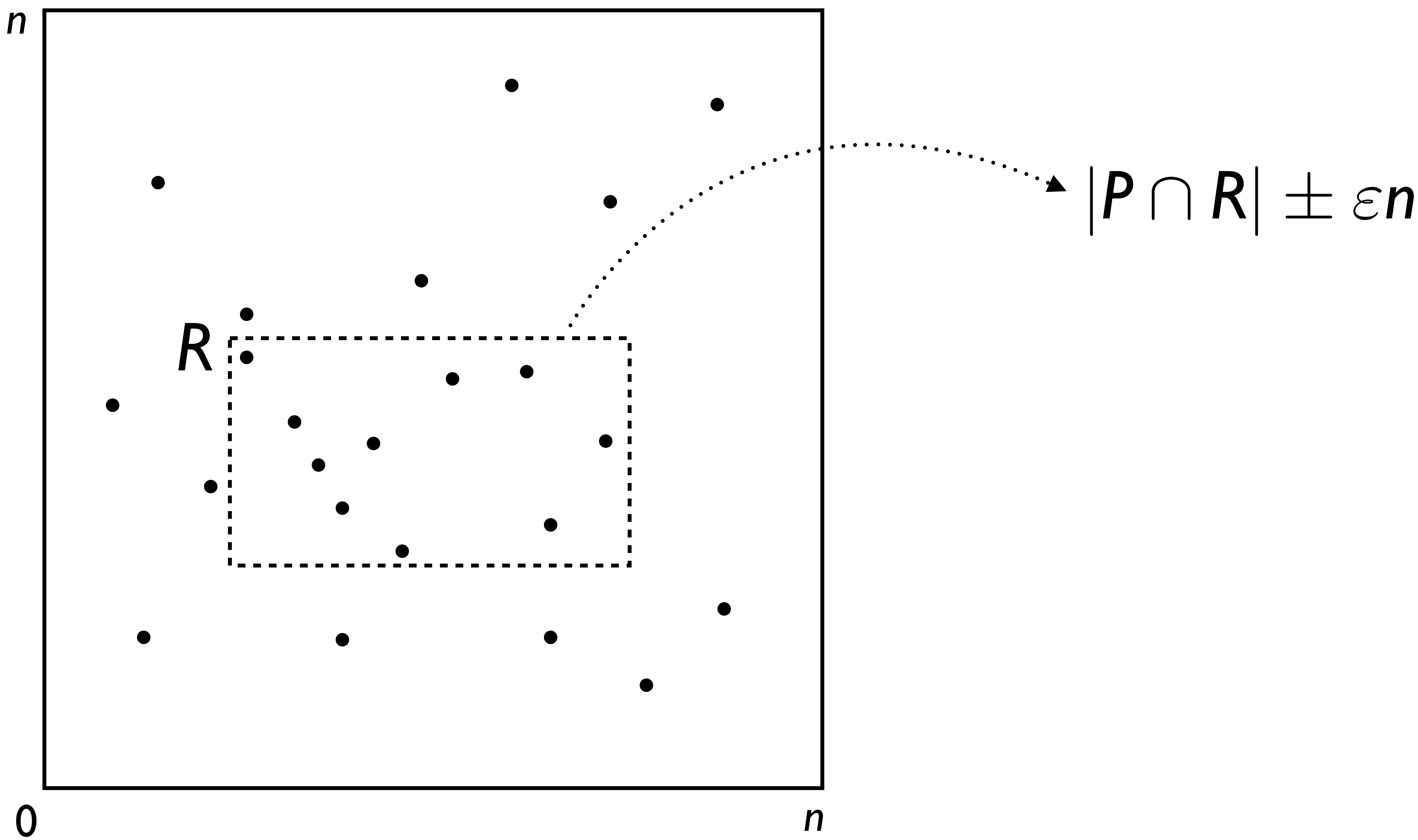
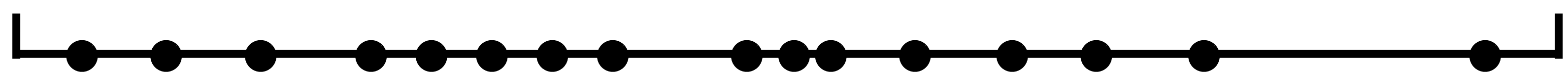


- Space complexity for a static data structure (summary)?

1-Dimensional Case

- Upperbound: $O(\frac{1}{\varepsilon} \log n)$ bits.



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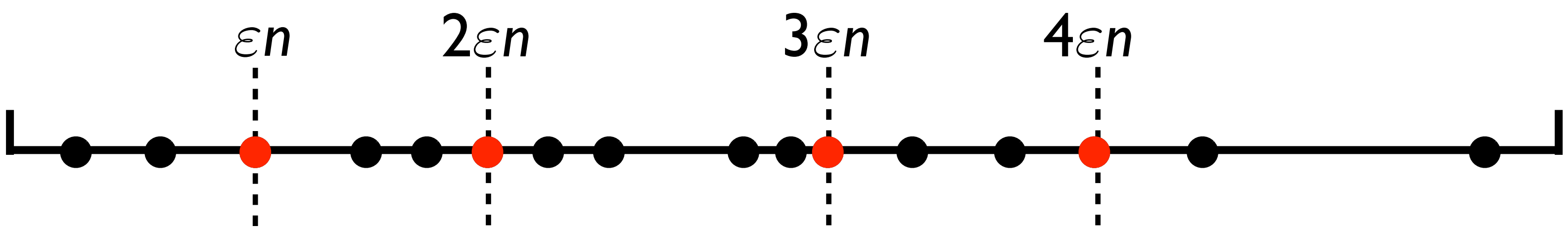

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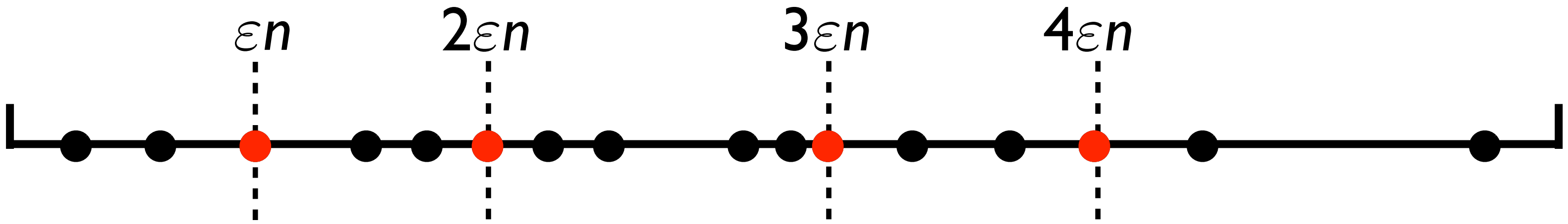

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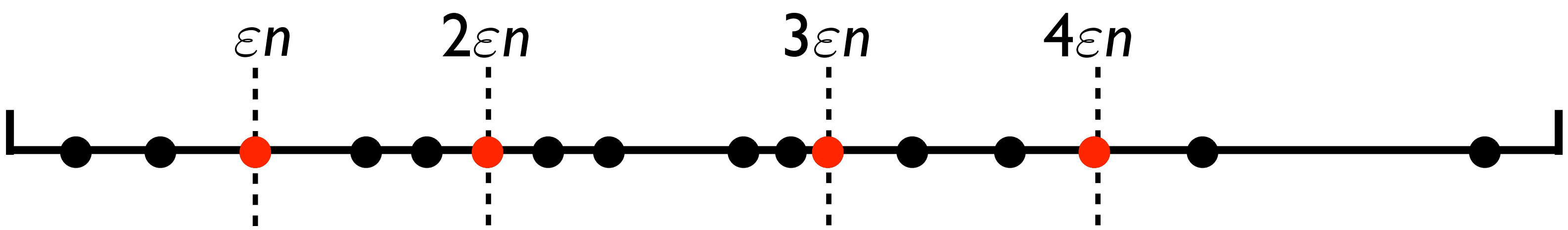

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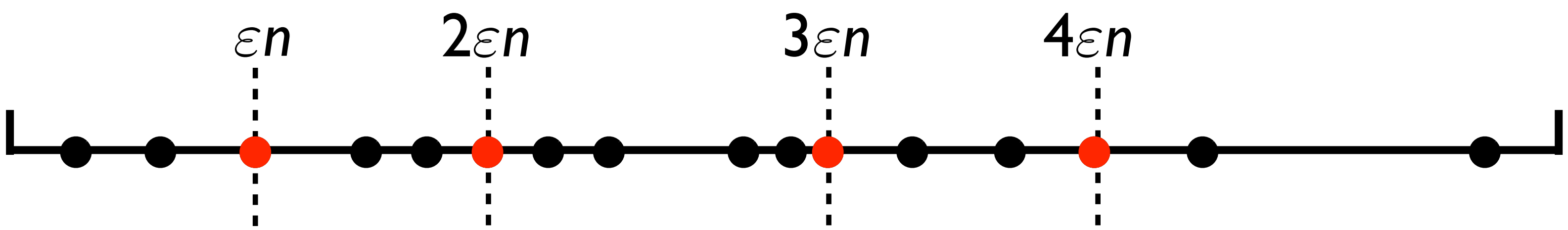

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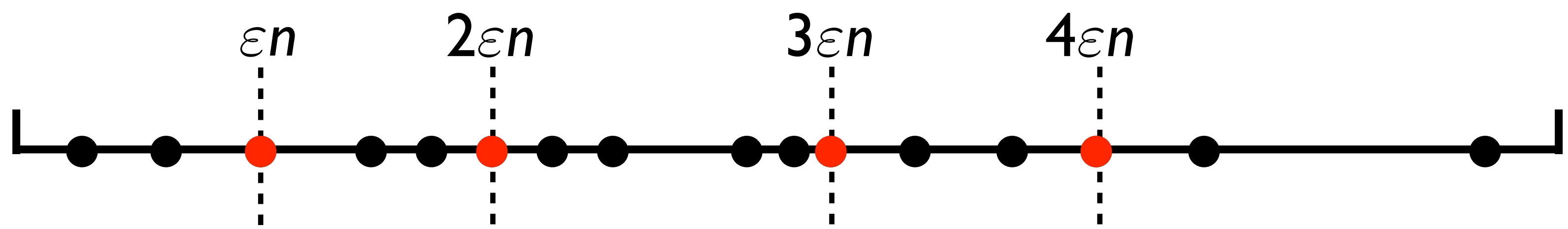

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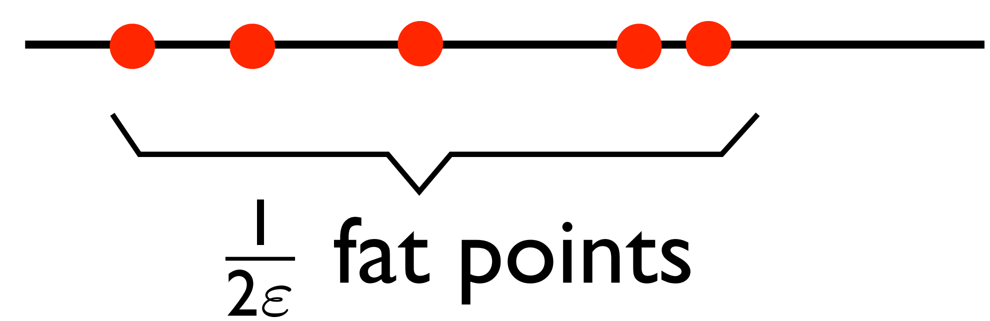

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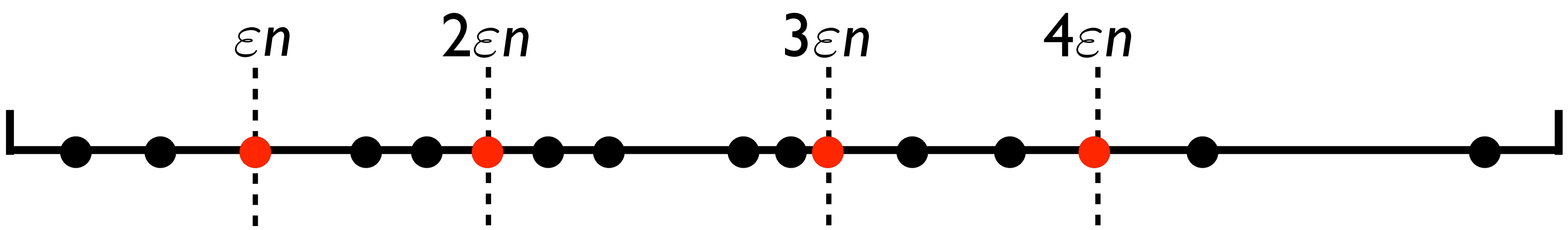

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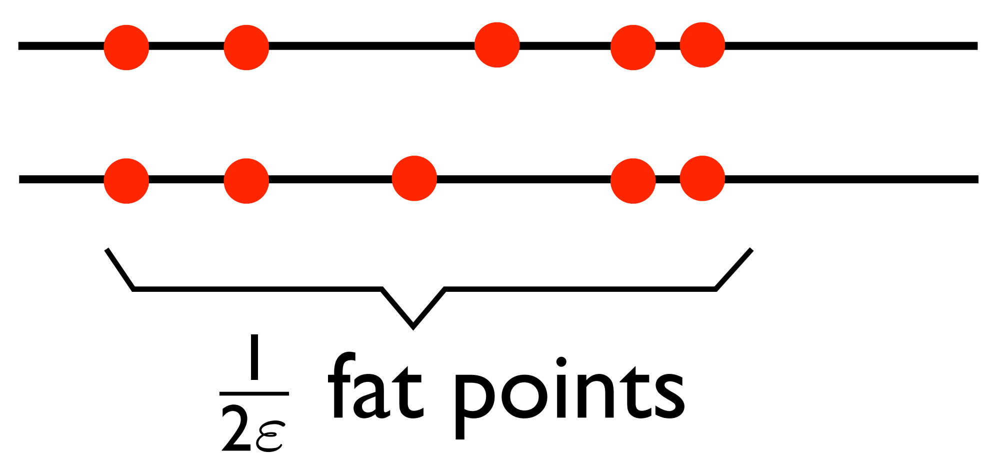

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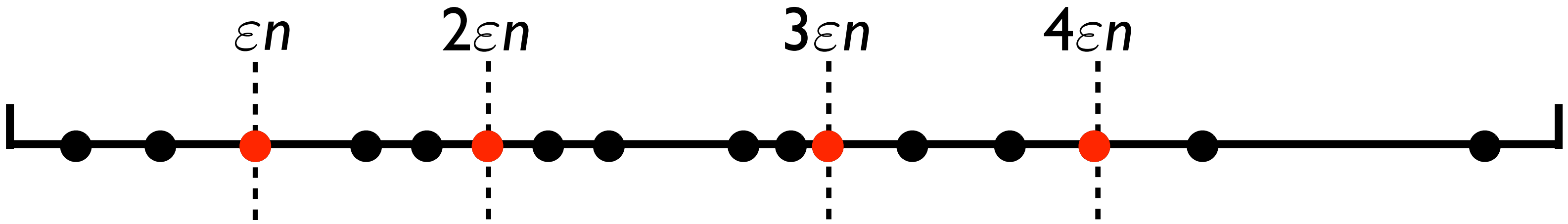

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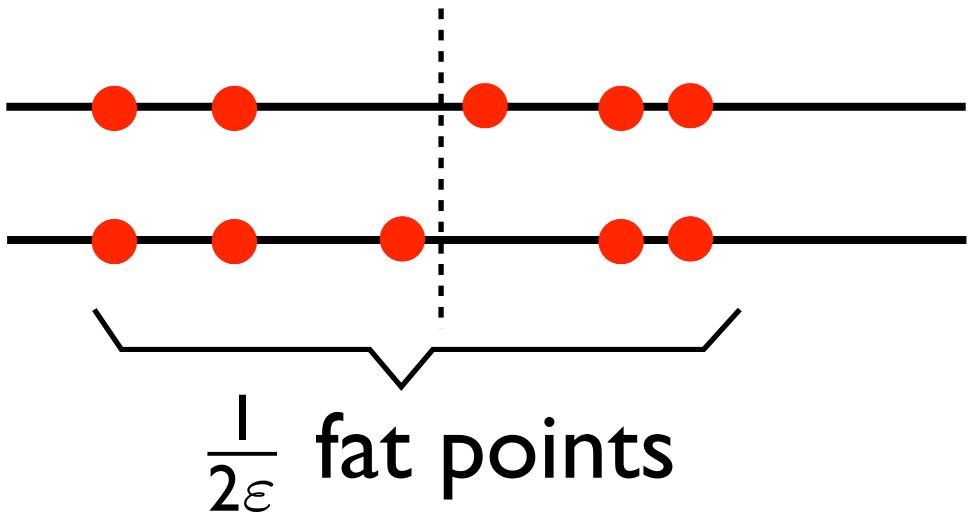

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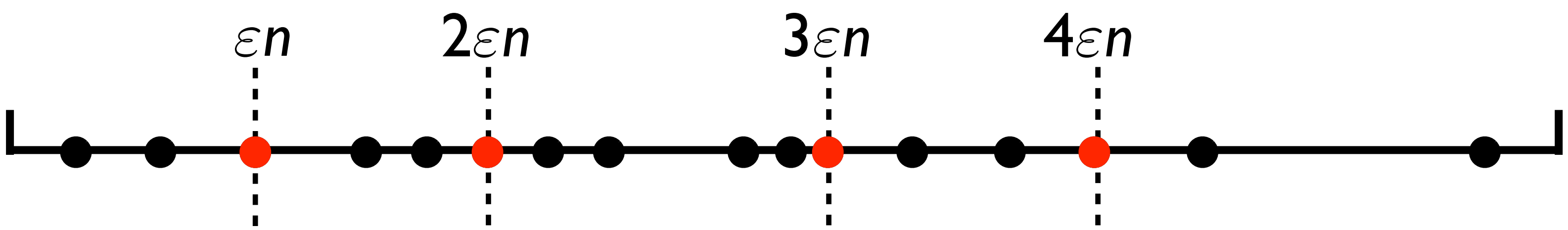

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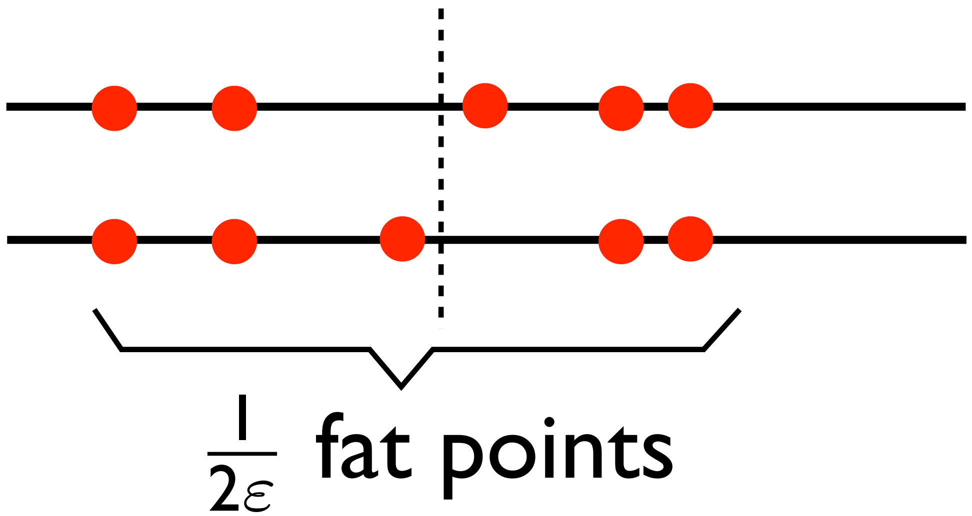

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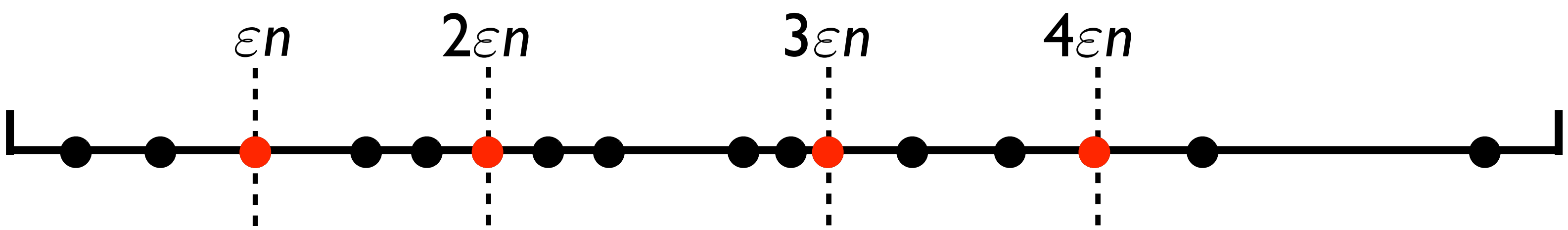

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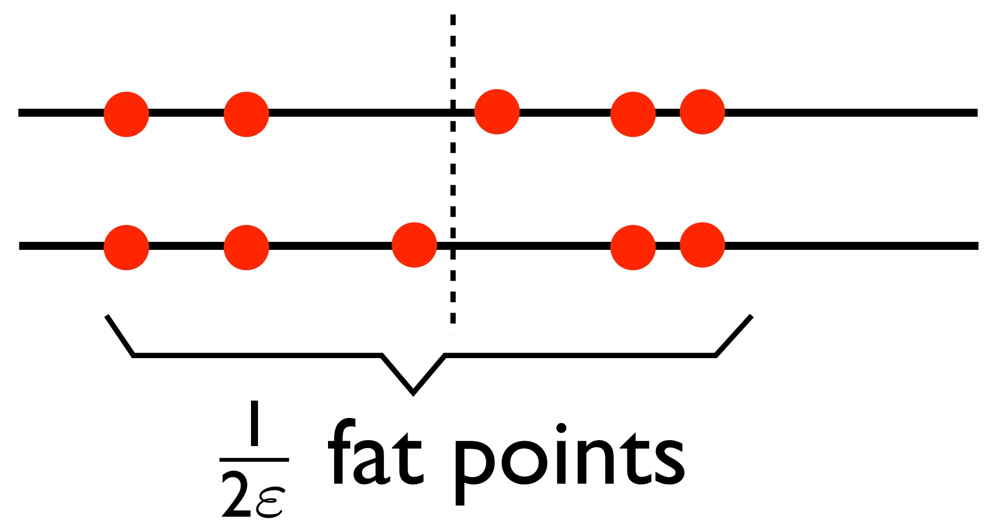

- $n^{\frac{1}{\varepsilon}}$ different point sets.
- $\Omega(\log n^{\frac{1}{\varepsilon}}) = \Omega(\frac{1}{\varepsilon}\log n)$ bits needed.

(Strong) Epsilon Approximation

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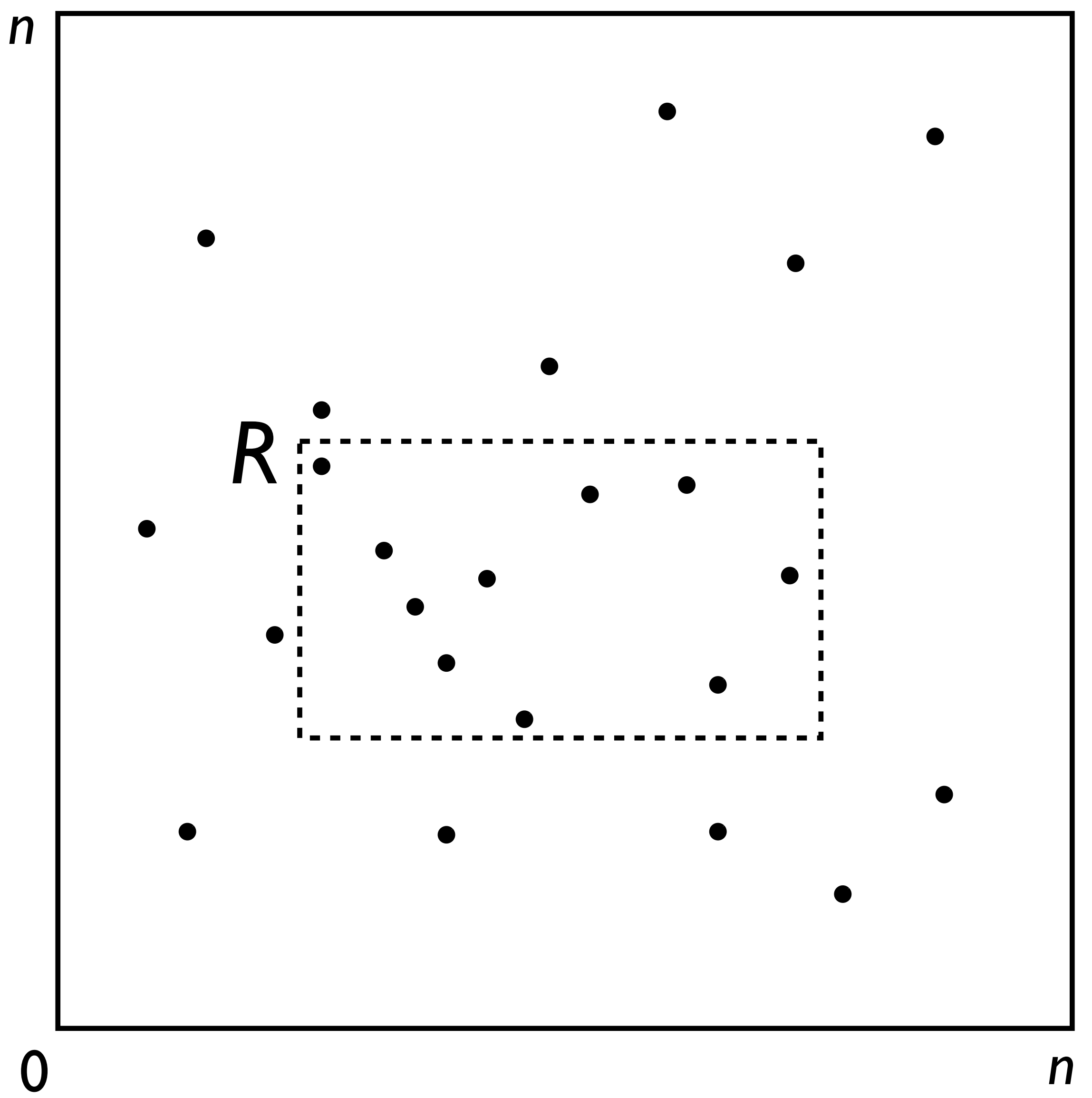

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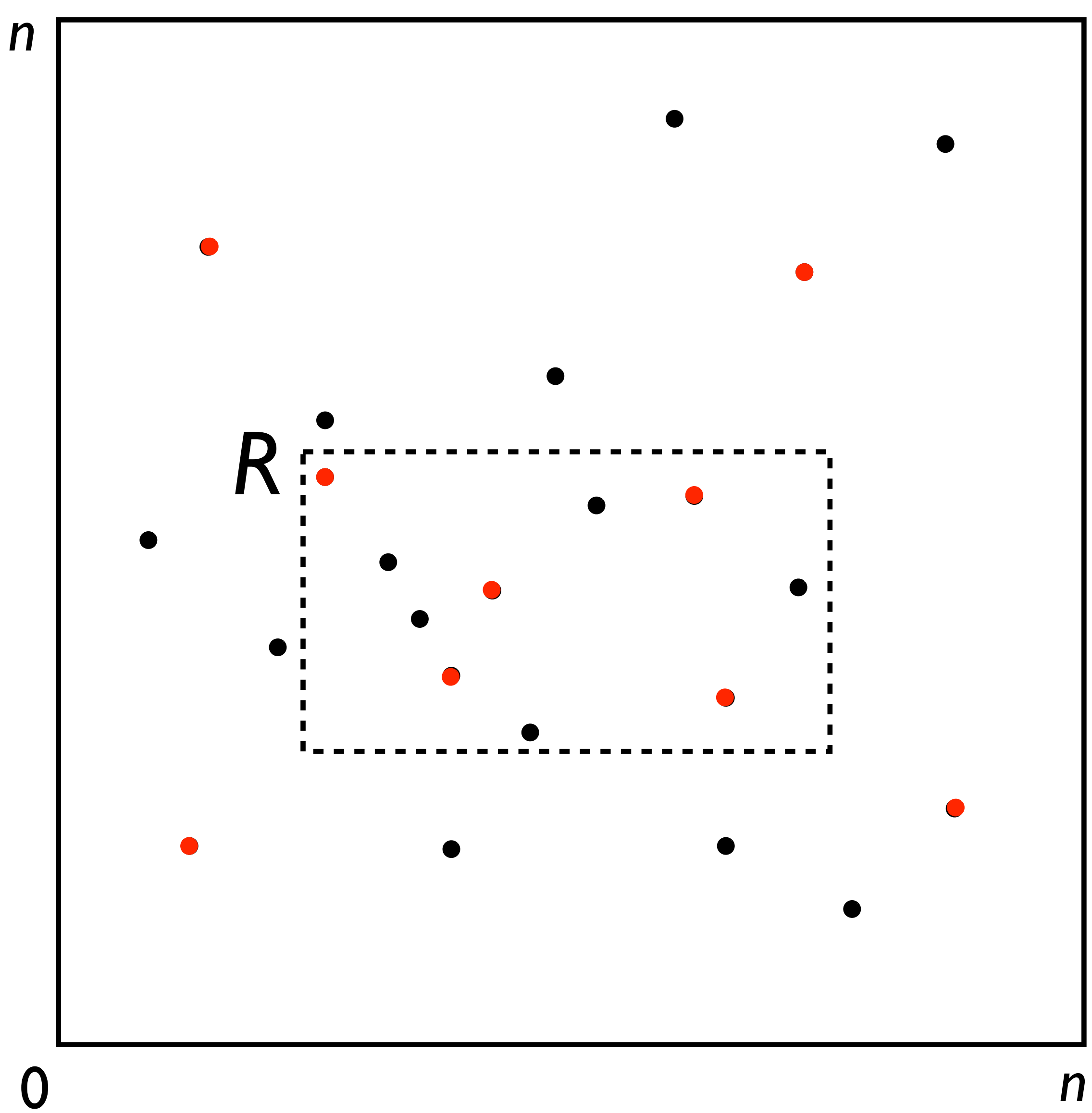

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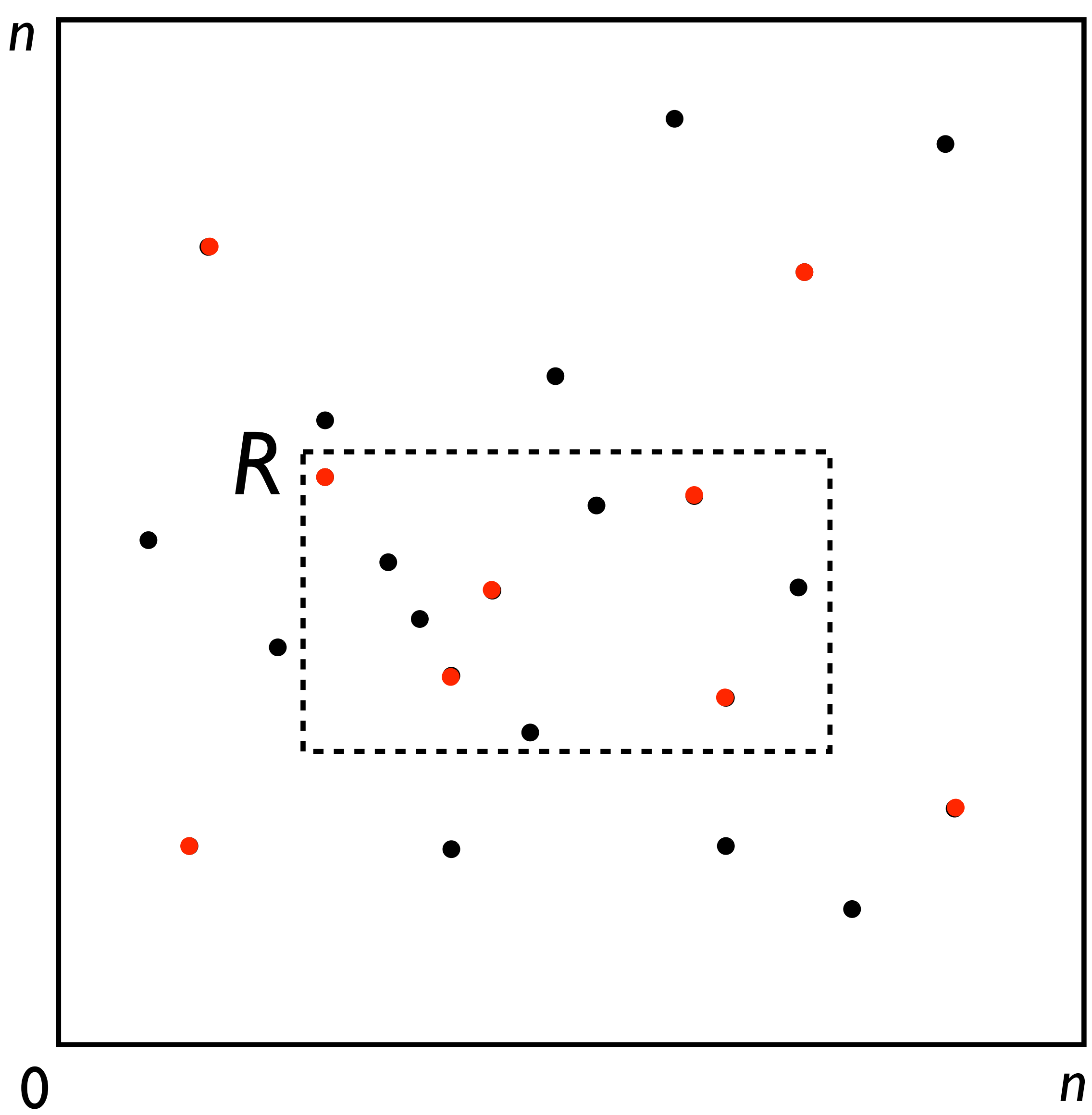

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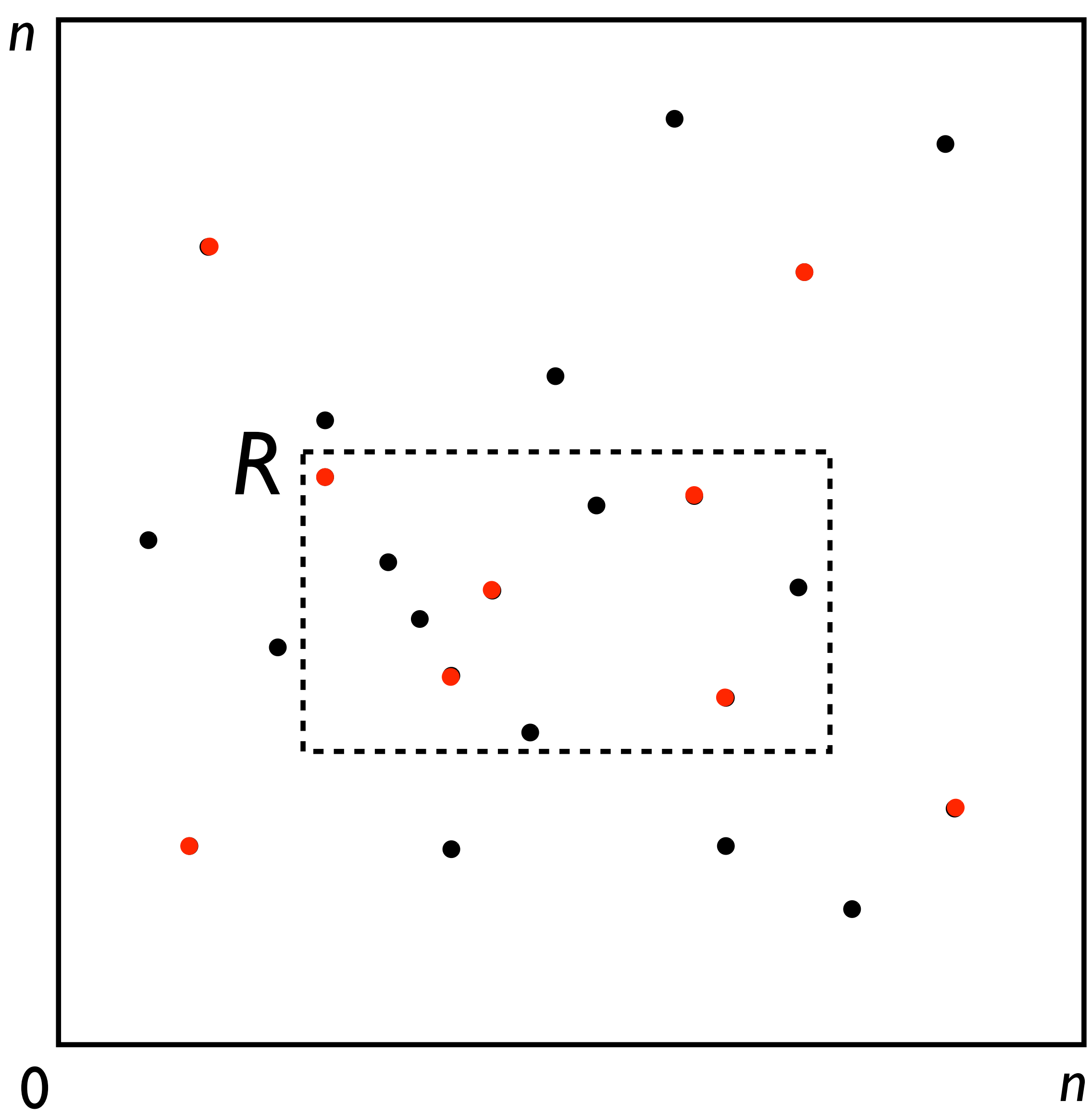

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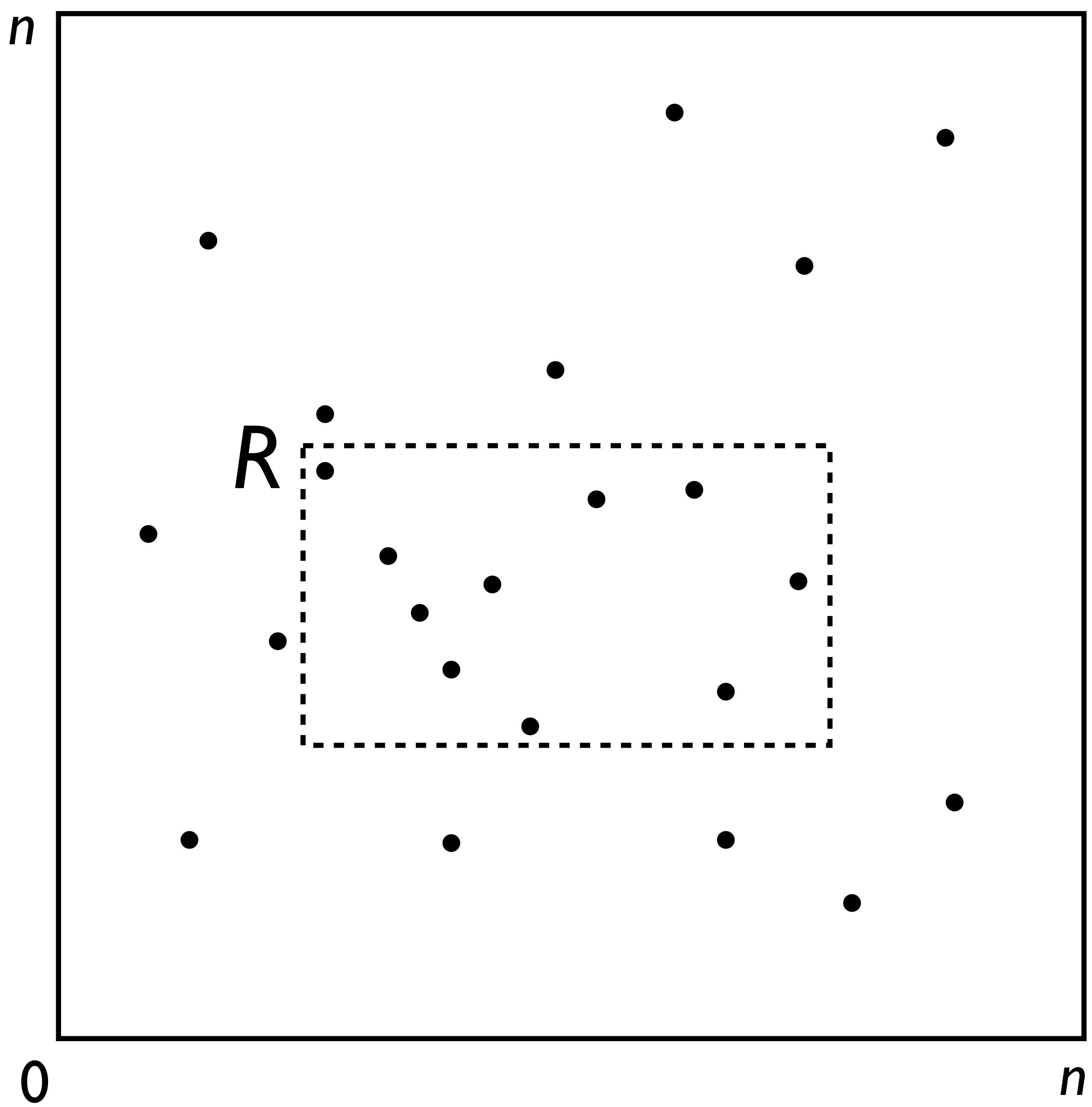

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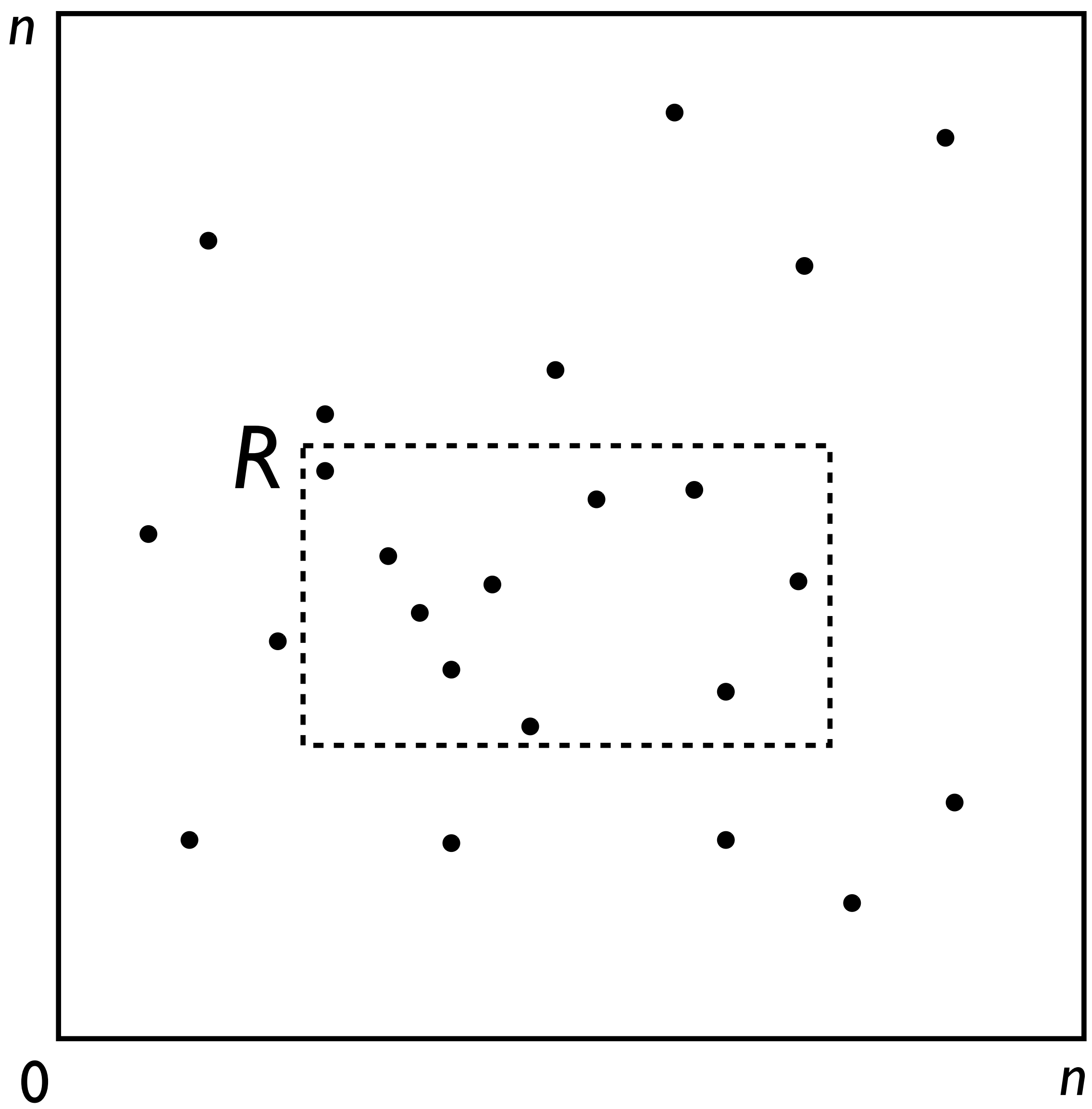

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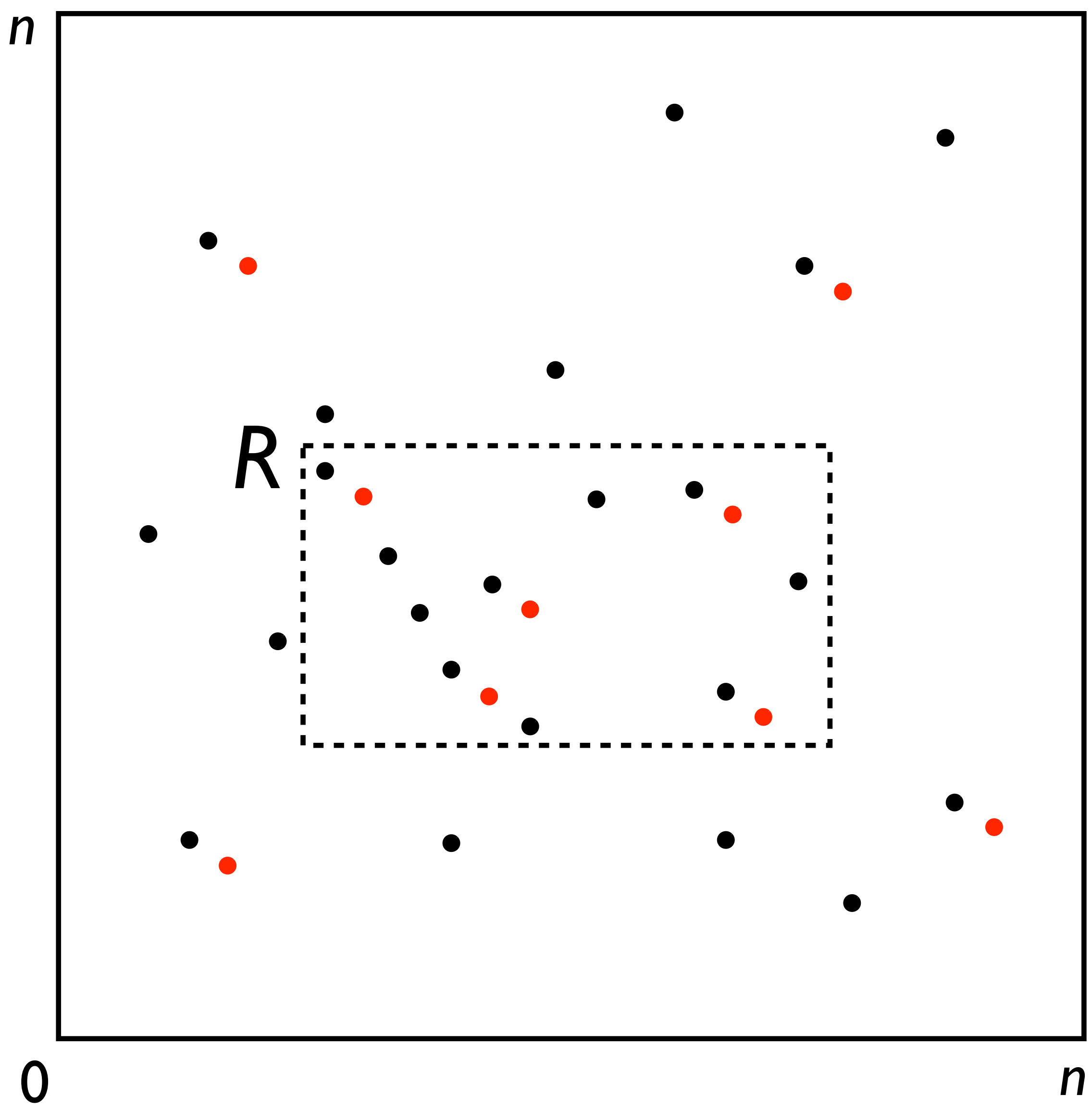

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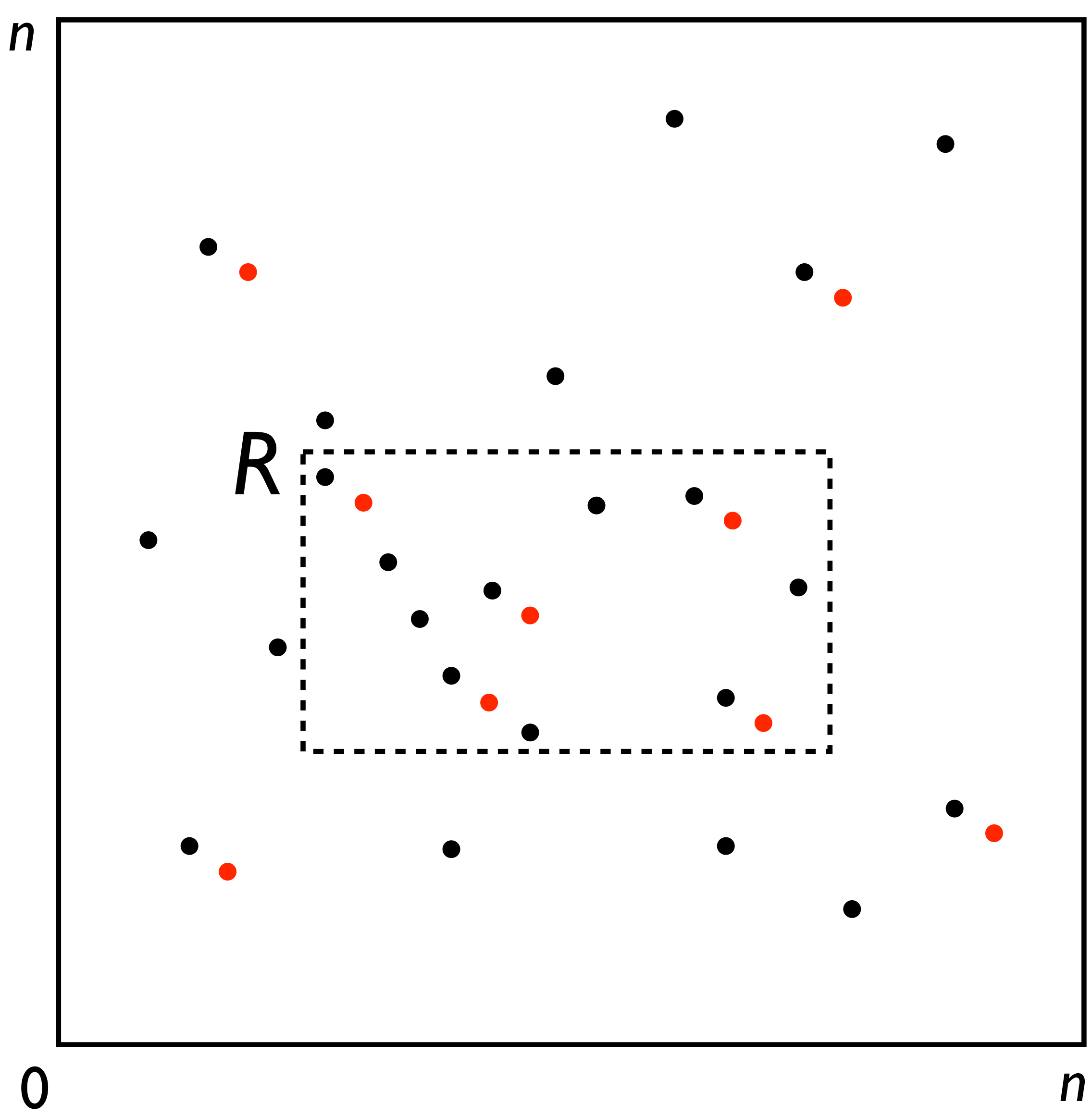

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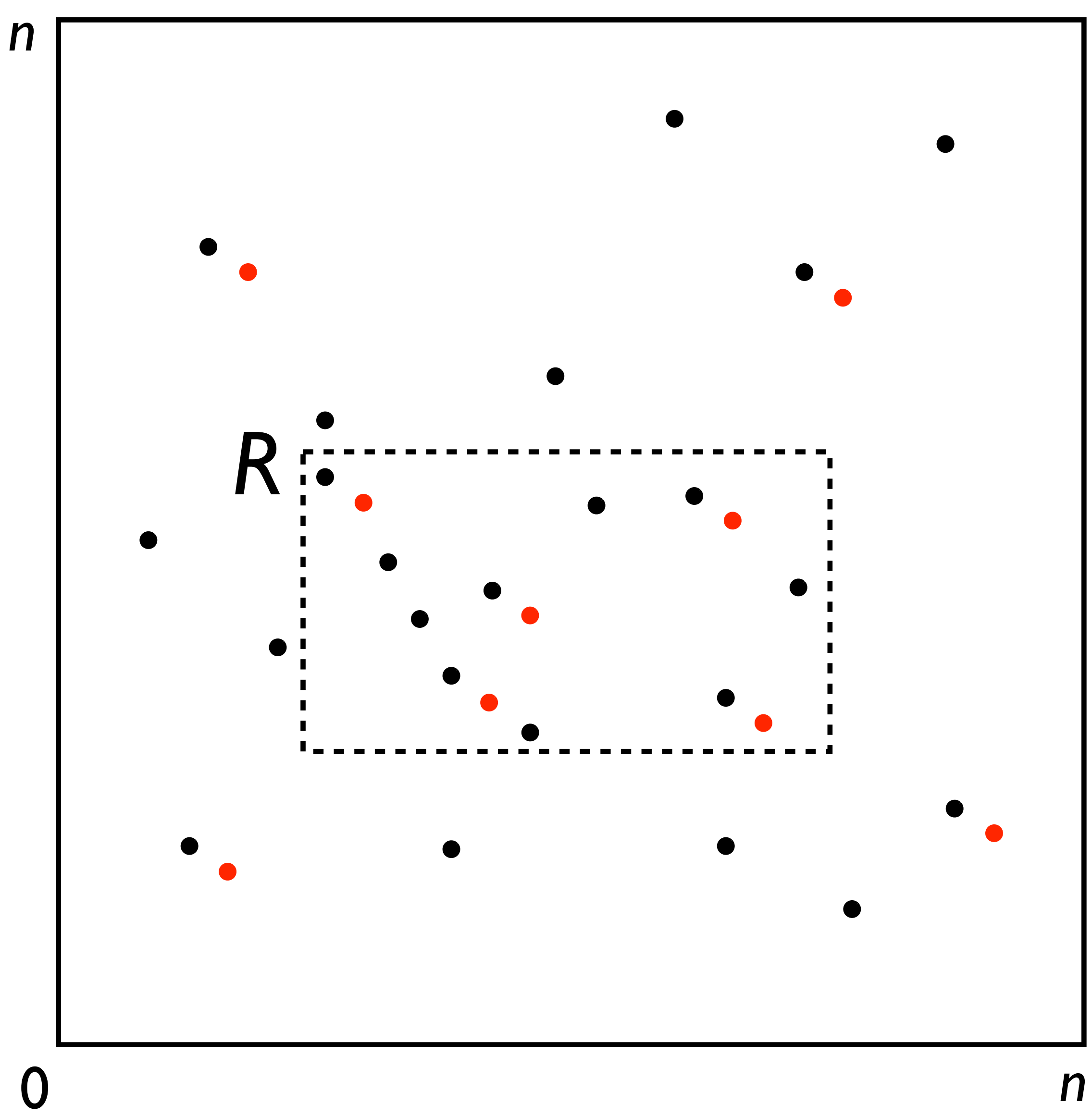

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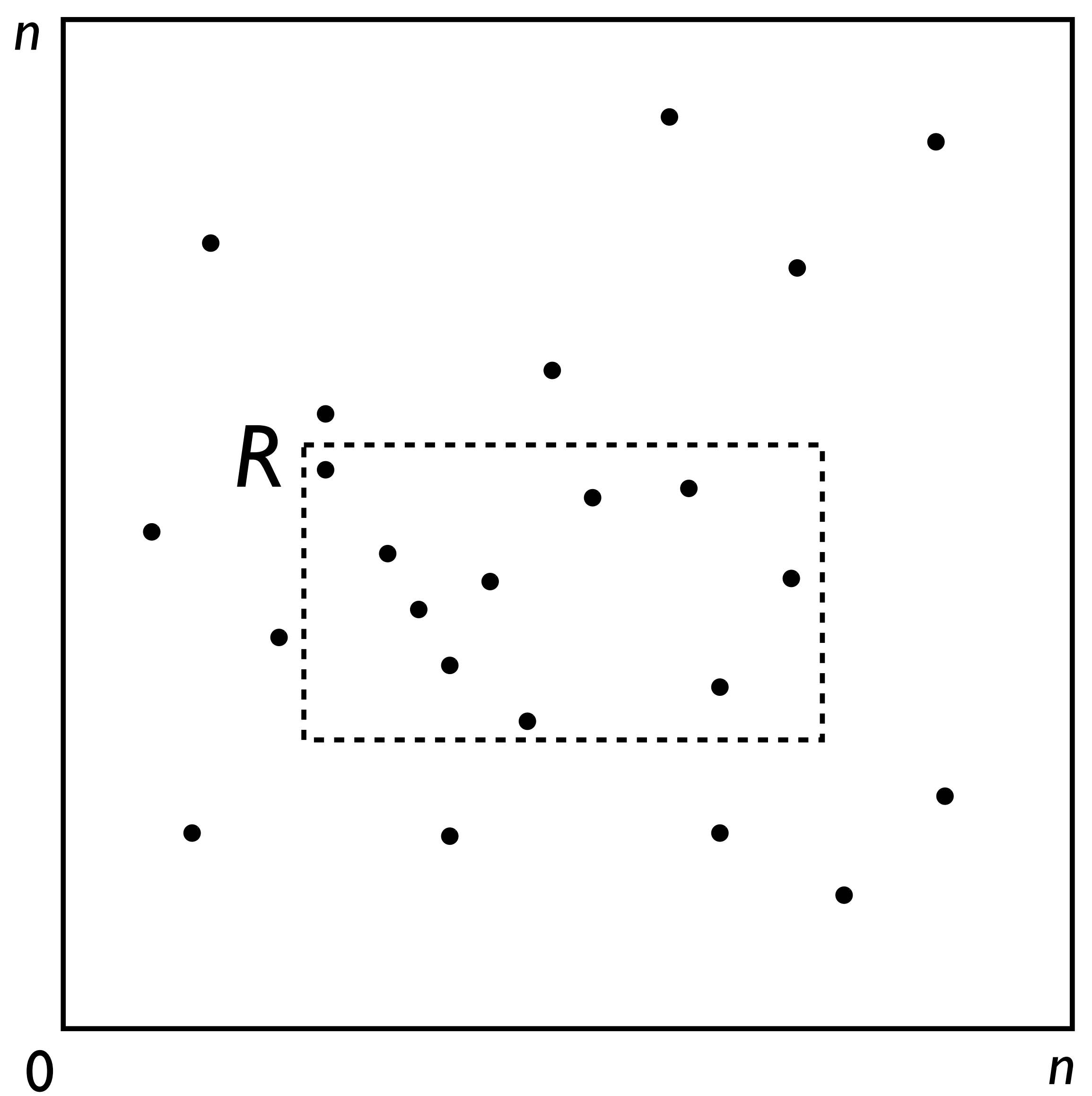
  - Orthogonal range counting with error $\log n$ is as hard as exact counting.

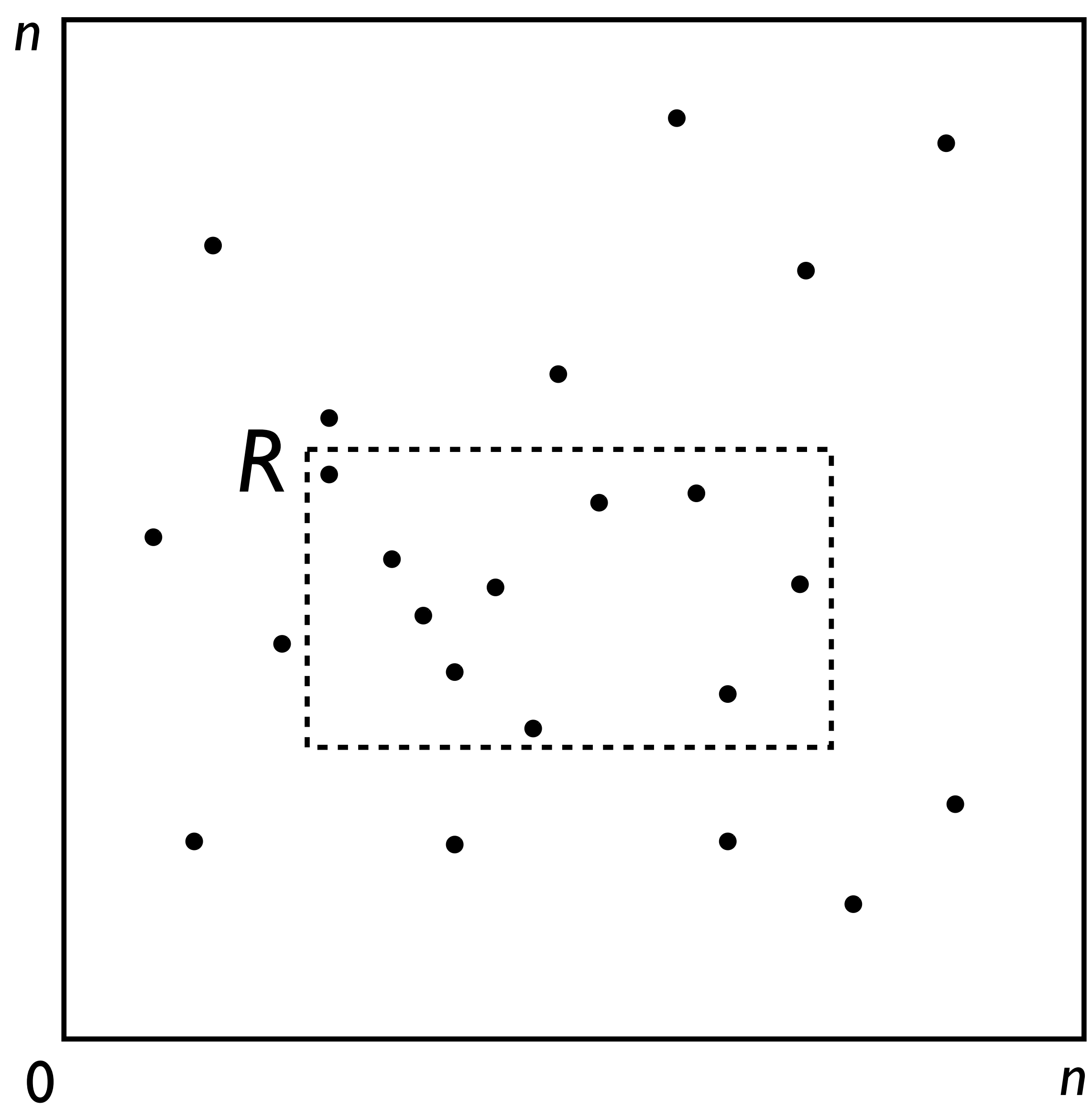
Preliminaries

Combinatorial Discrepancy



Combinatorial Discrepancy

- Range space $(P, \mathcal{R})$ and a color function $\chi : P \rightarrow \{-1, +1\}$.



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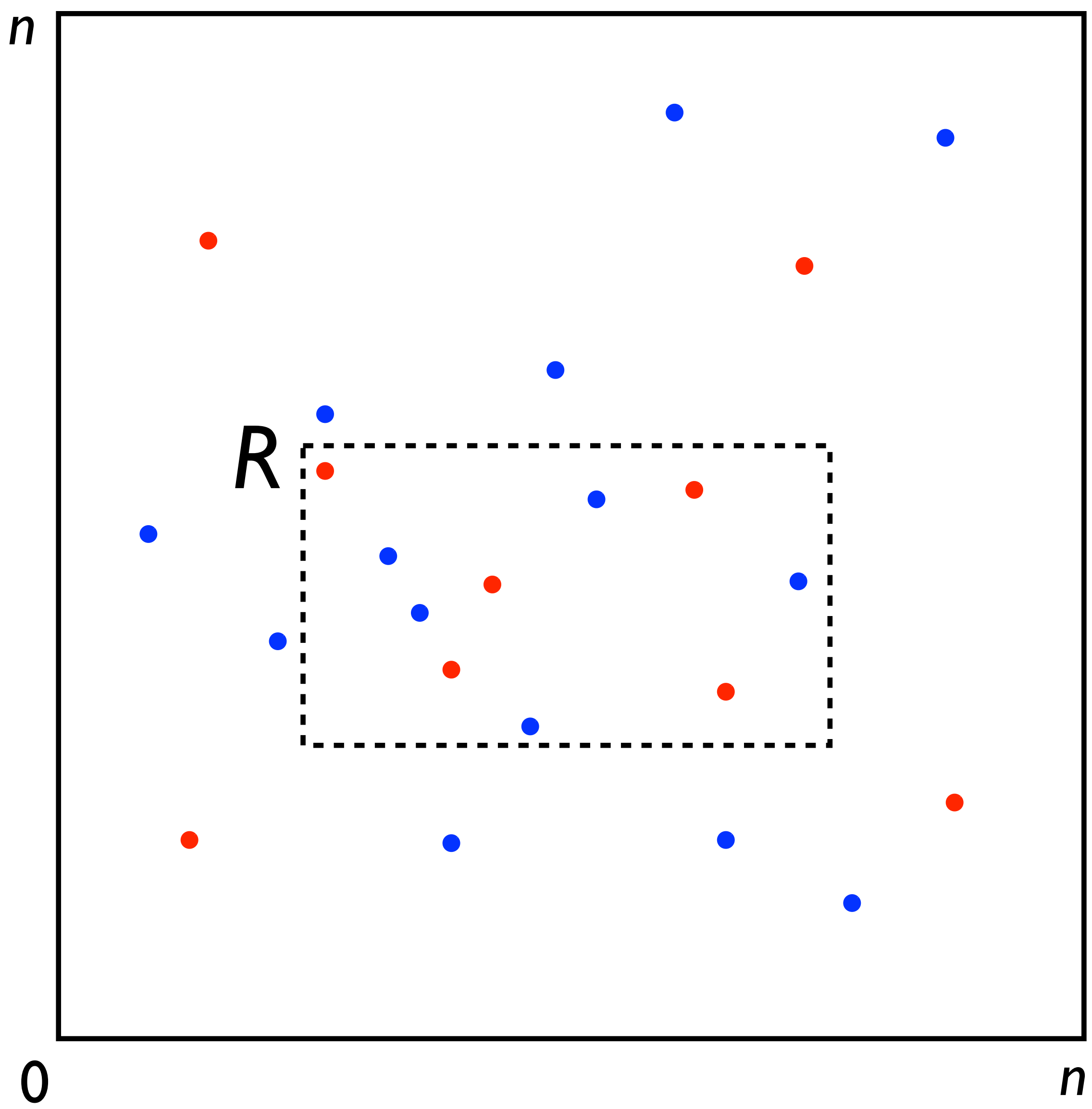

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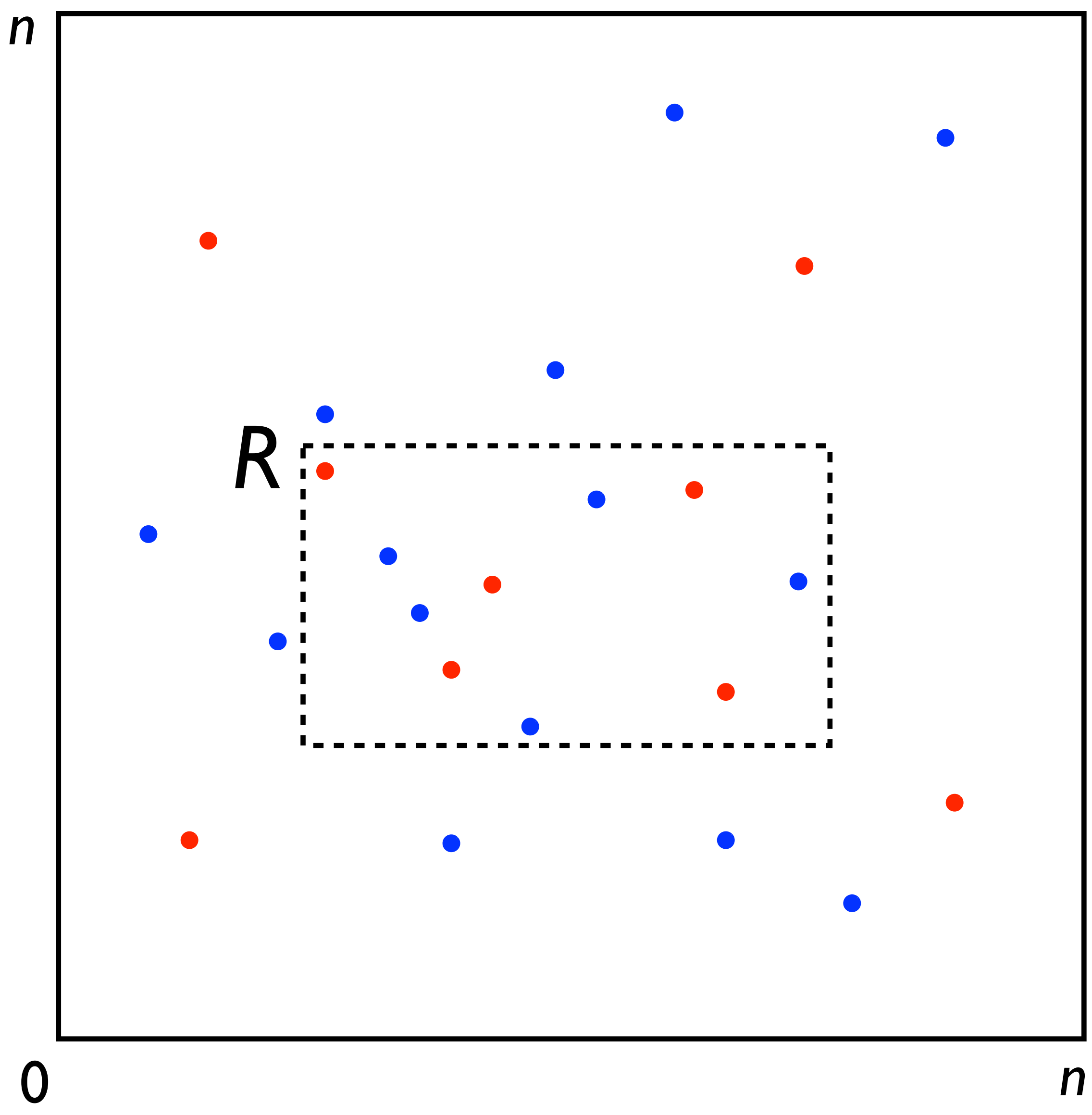

$$\chi(P \cap R) = \sum_{p \in P \cap R} \chi(p);$$

$$\mathrm{disc}(P, \mathcal{R}) = \min_{\chi} \max_{R \in \mathcal{R}} |\chi(P \cap R)|;$$

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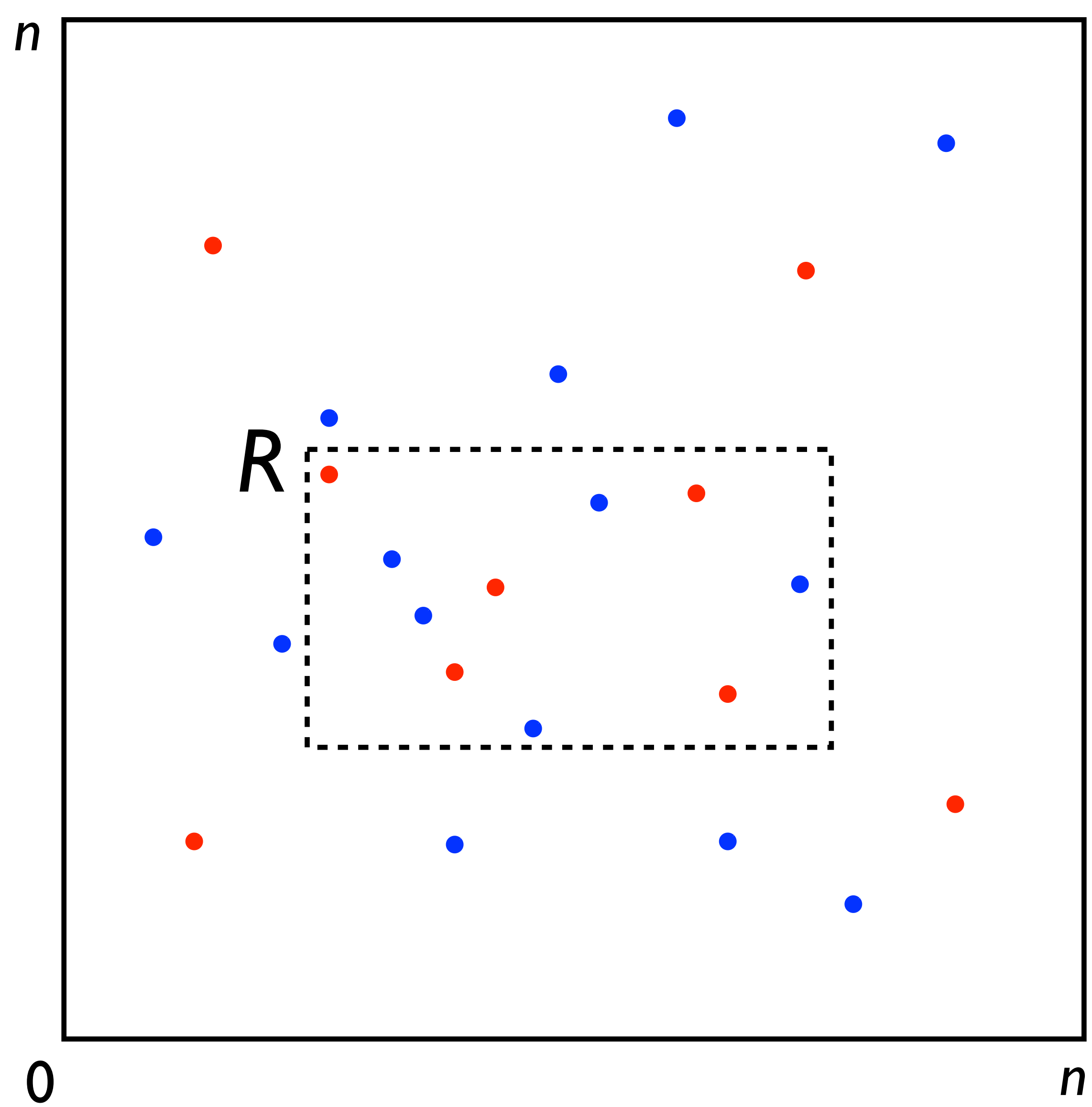

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- Upperbound: $O(\log^{2.5} n)$ (Babsal 2010); Lowerbound: $\Omega(\log n)$ (Beck 1981).

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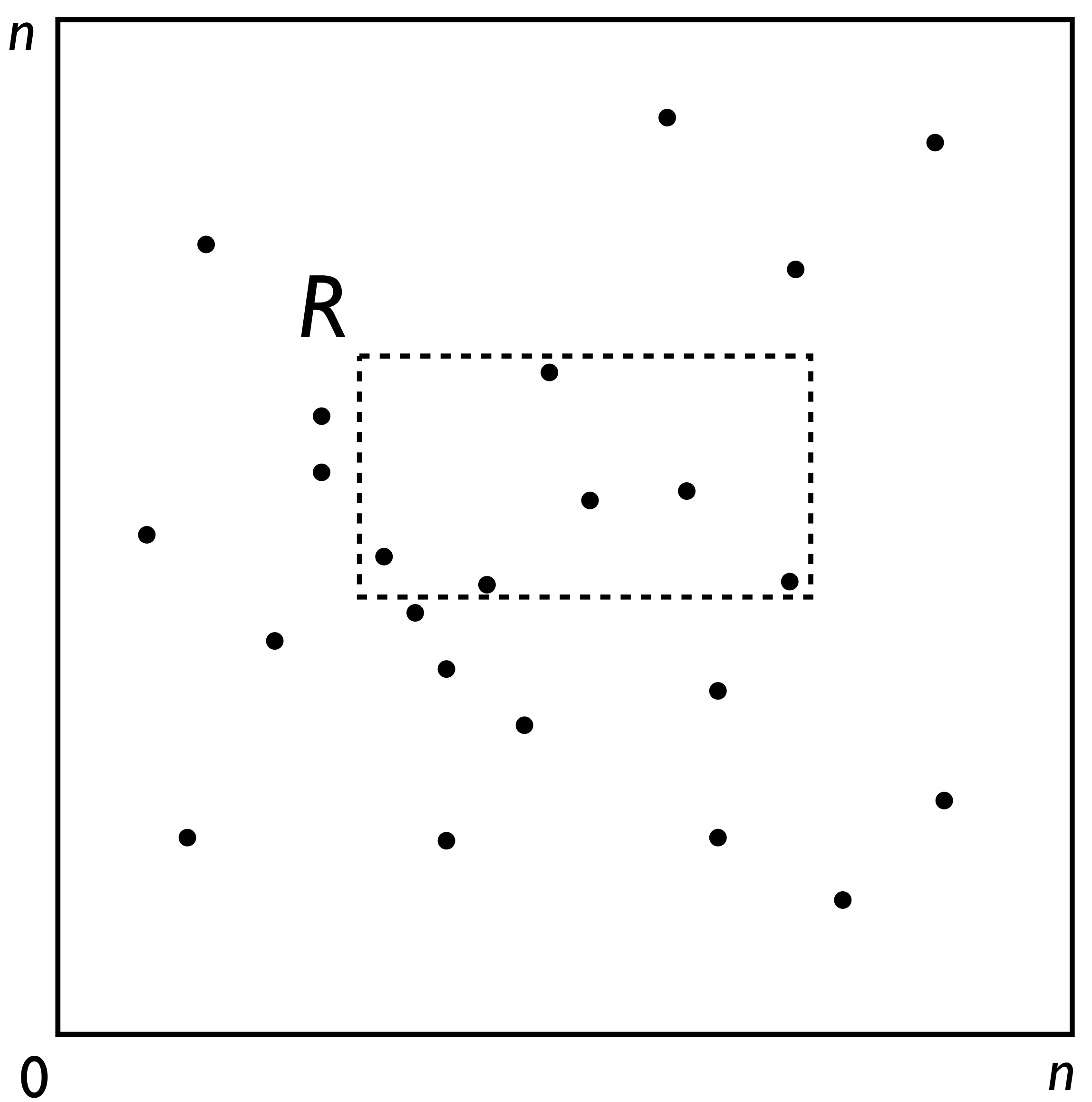
  - $D(n, \mathcal{R}) = \sup_{|P|=n} D(P, \mathcal{R})$.

- Upper bound & Lower bound: $\Theta(\log n)$ (Schmidt 1972).

(Strong) Epsilon Net

- N: A subset of P.



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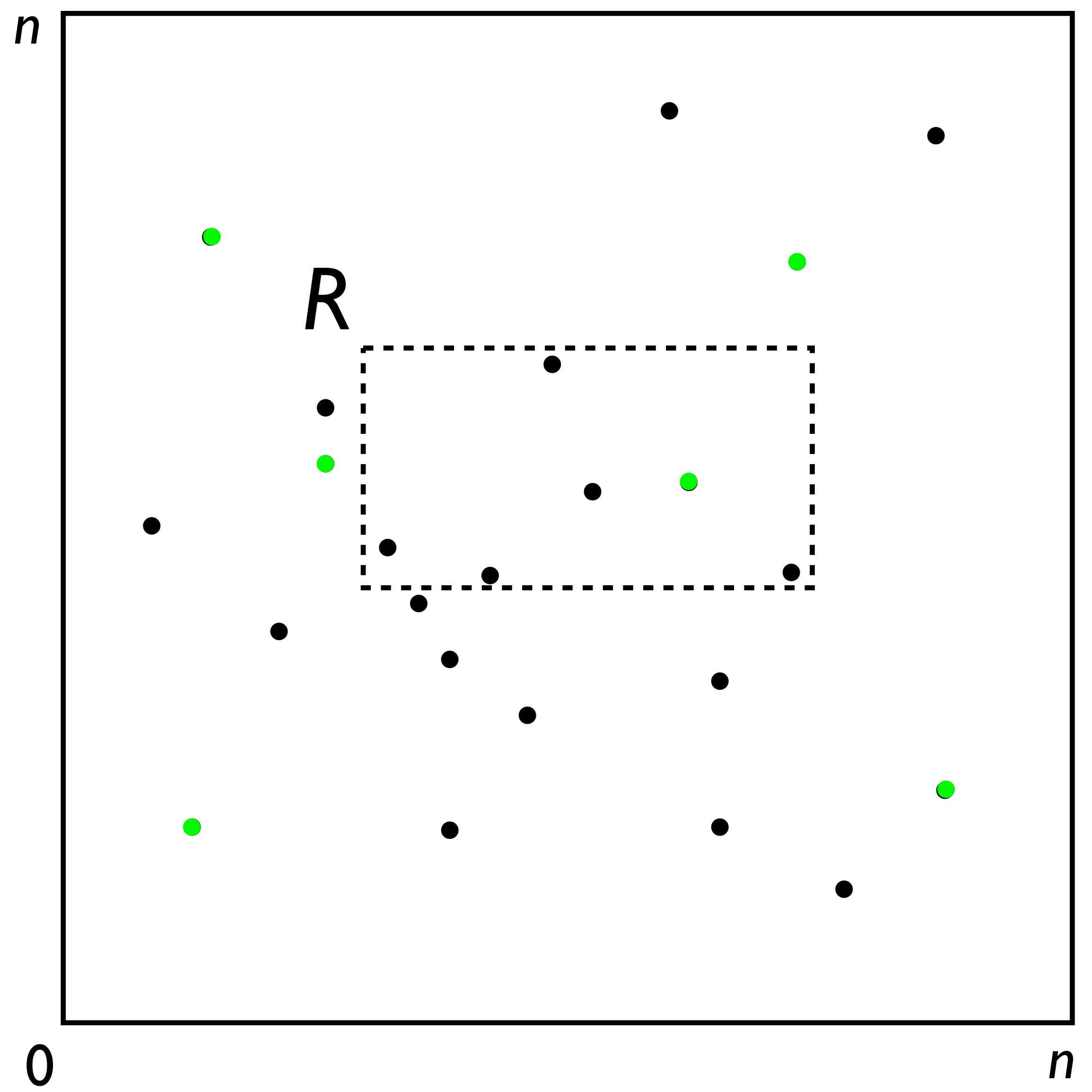

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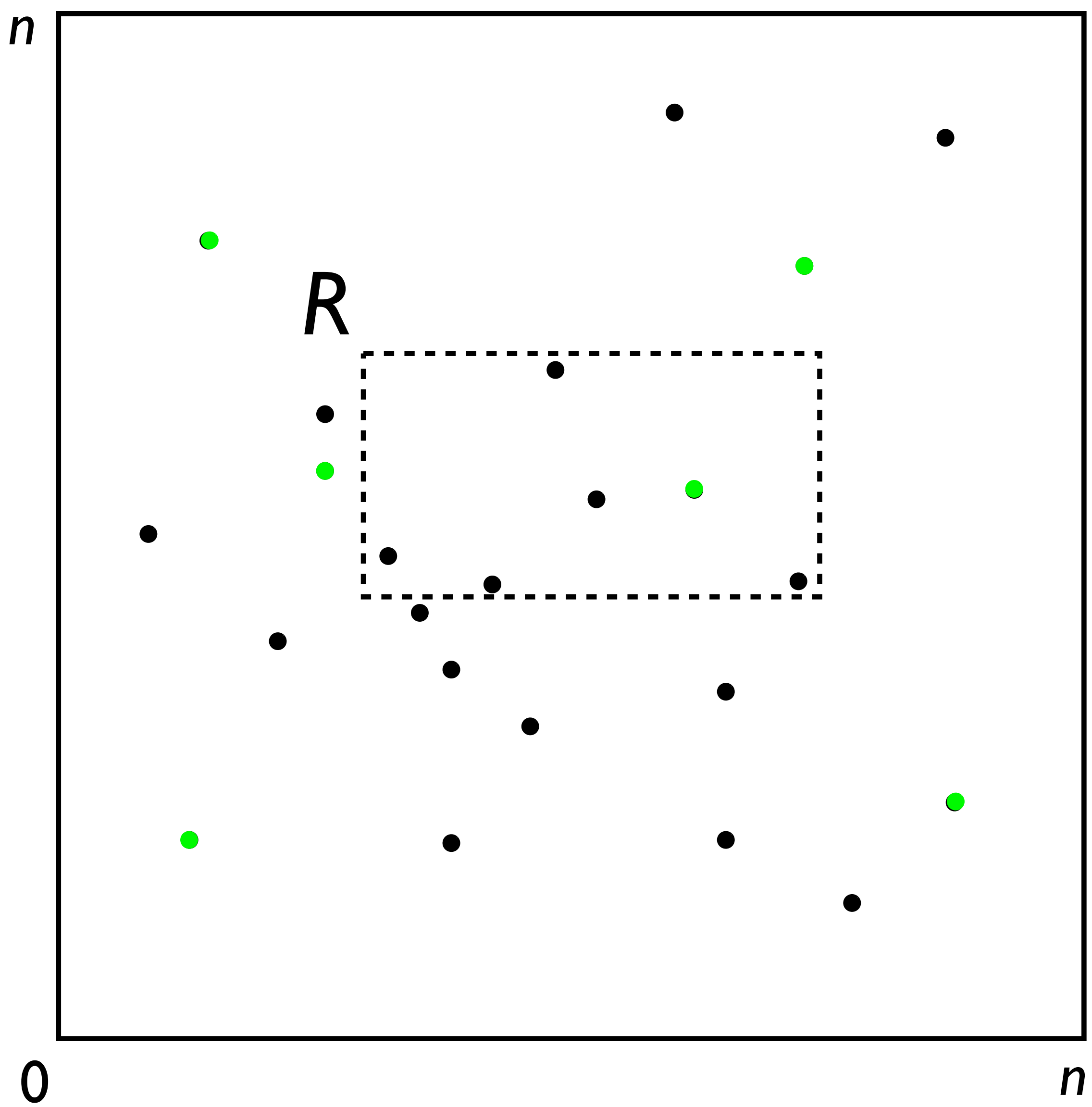

If $|R \cap P| \geq \varepsilon n$

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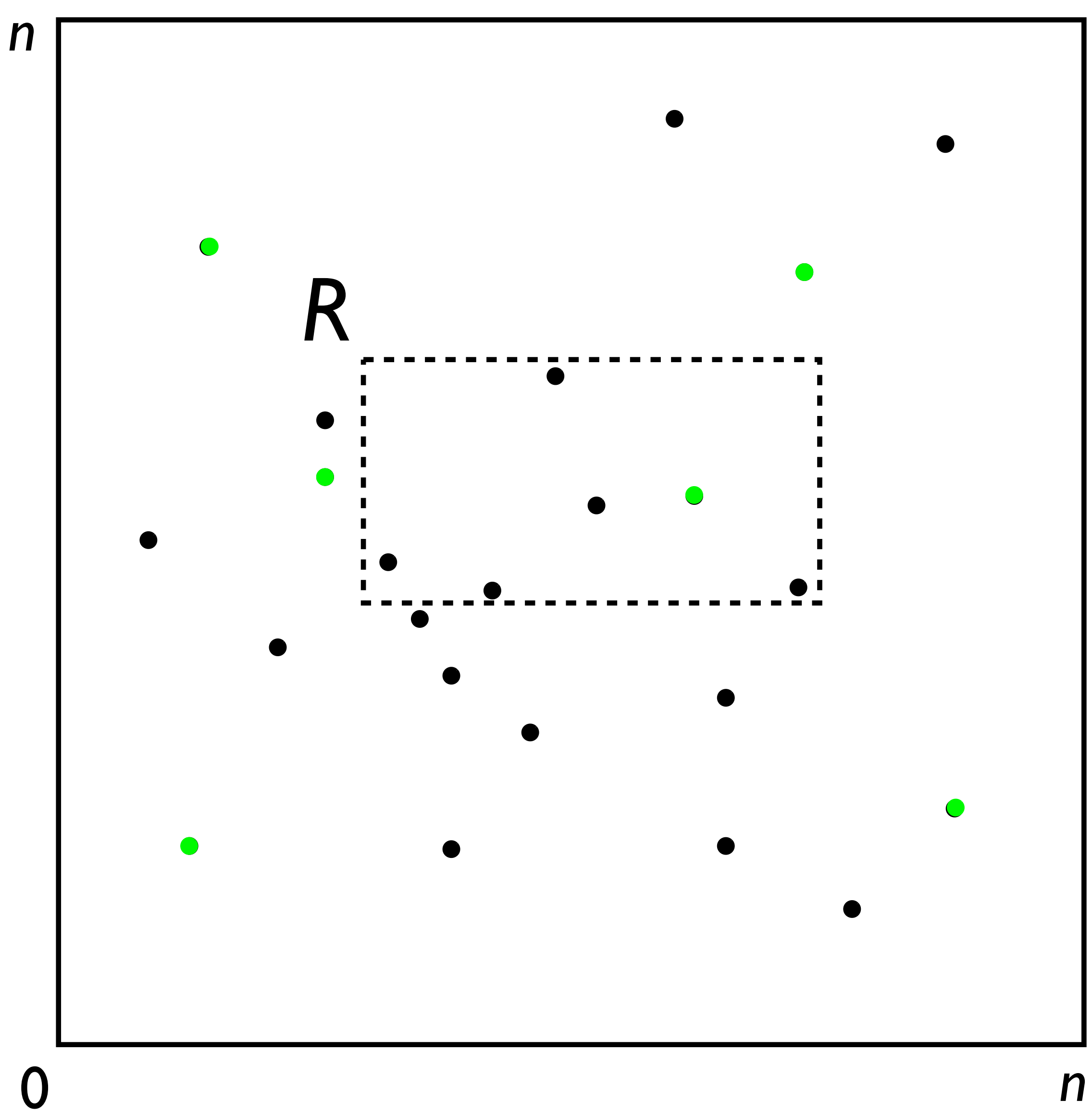

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- $\Theta(\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon})$ points (Aronv, et.al. 2010, Pach and Tardos 2011).

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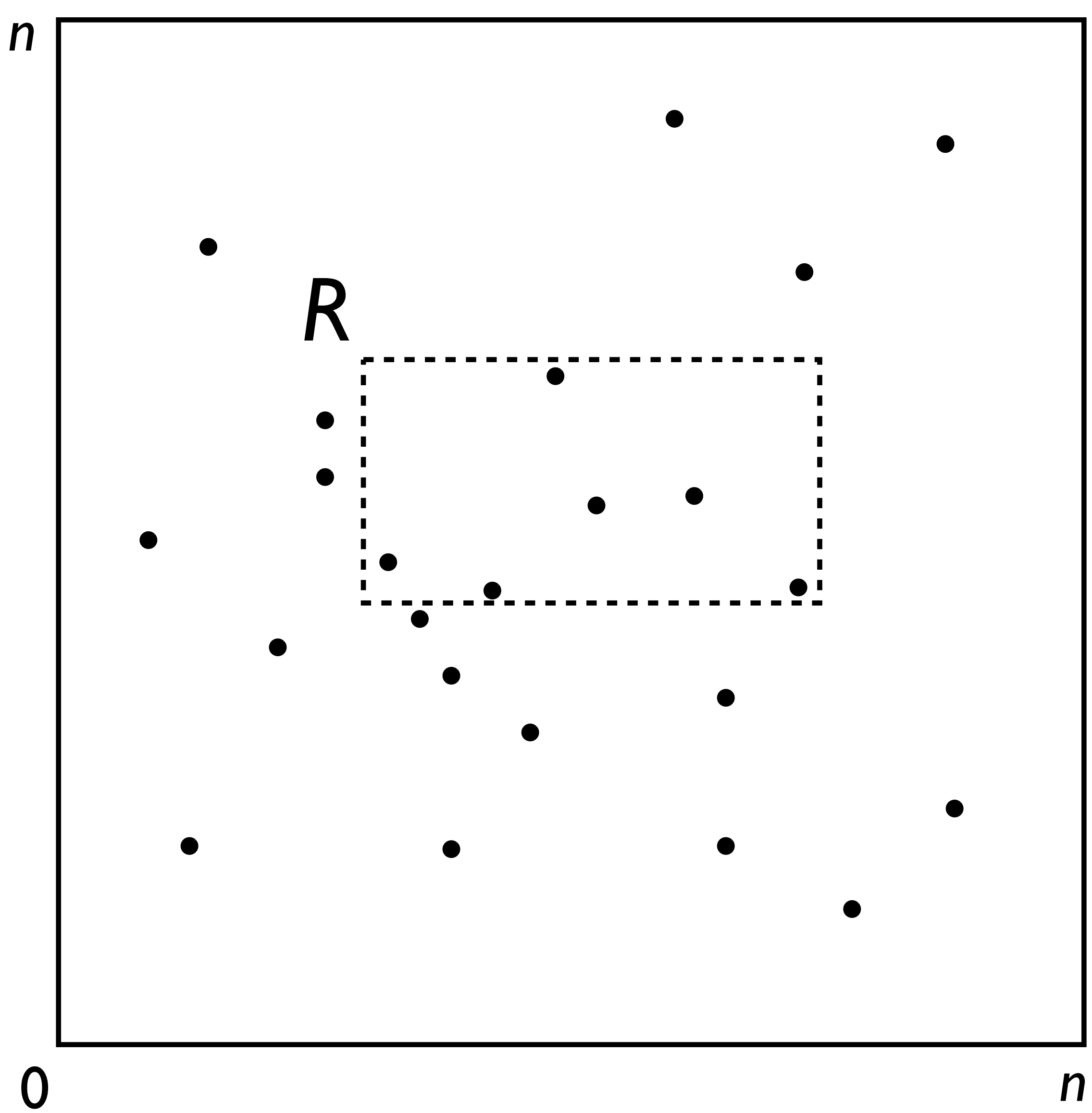

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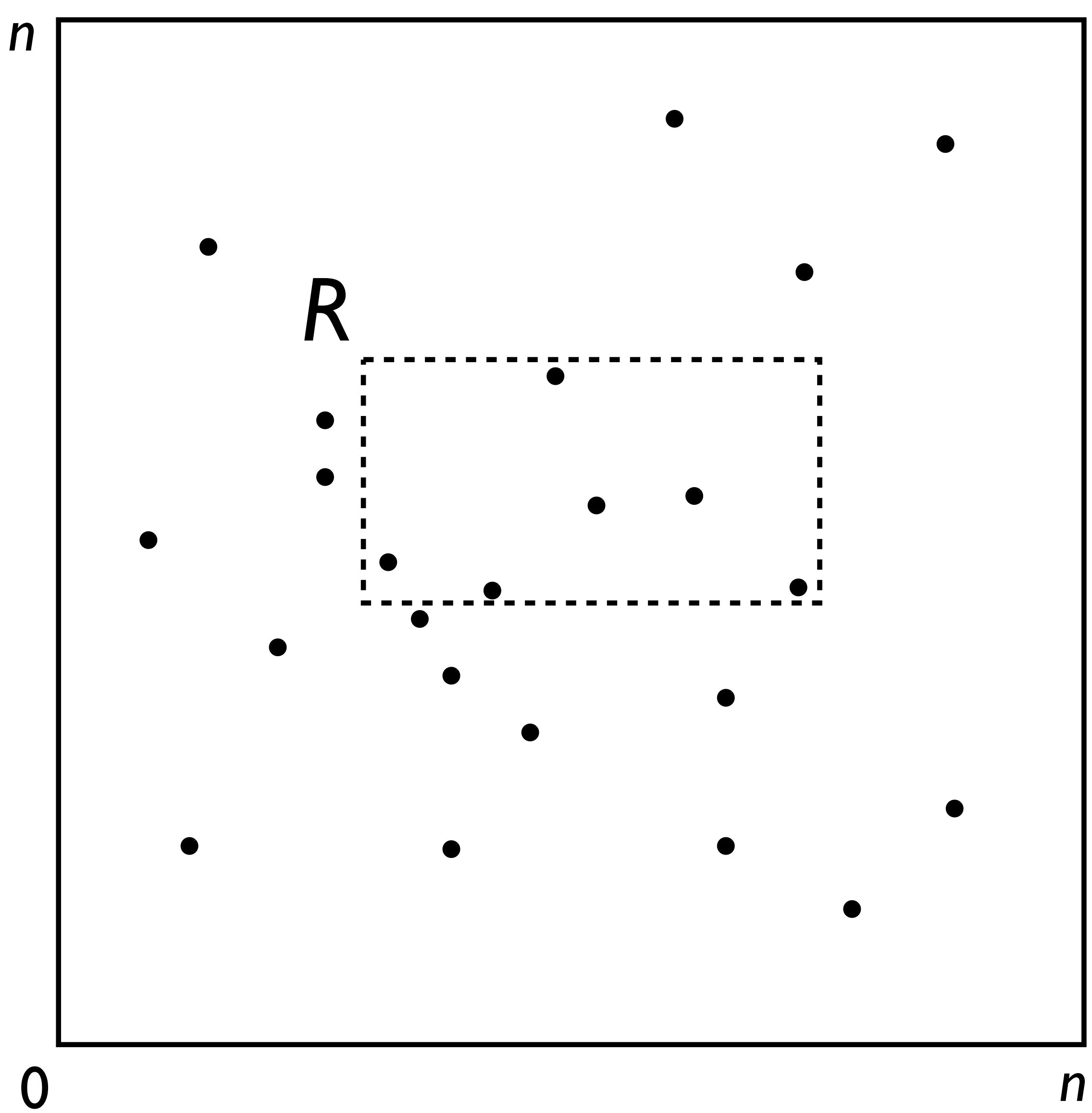

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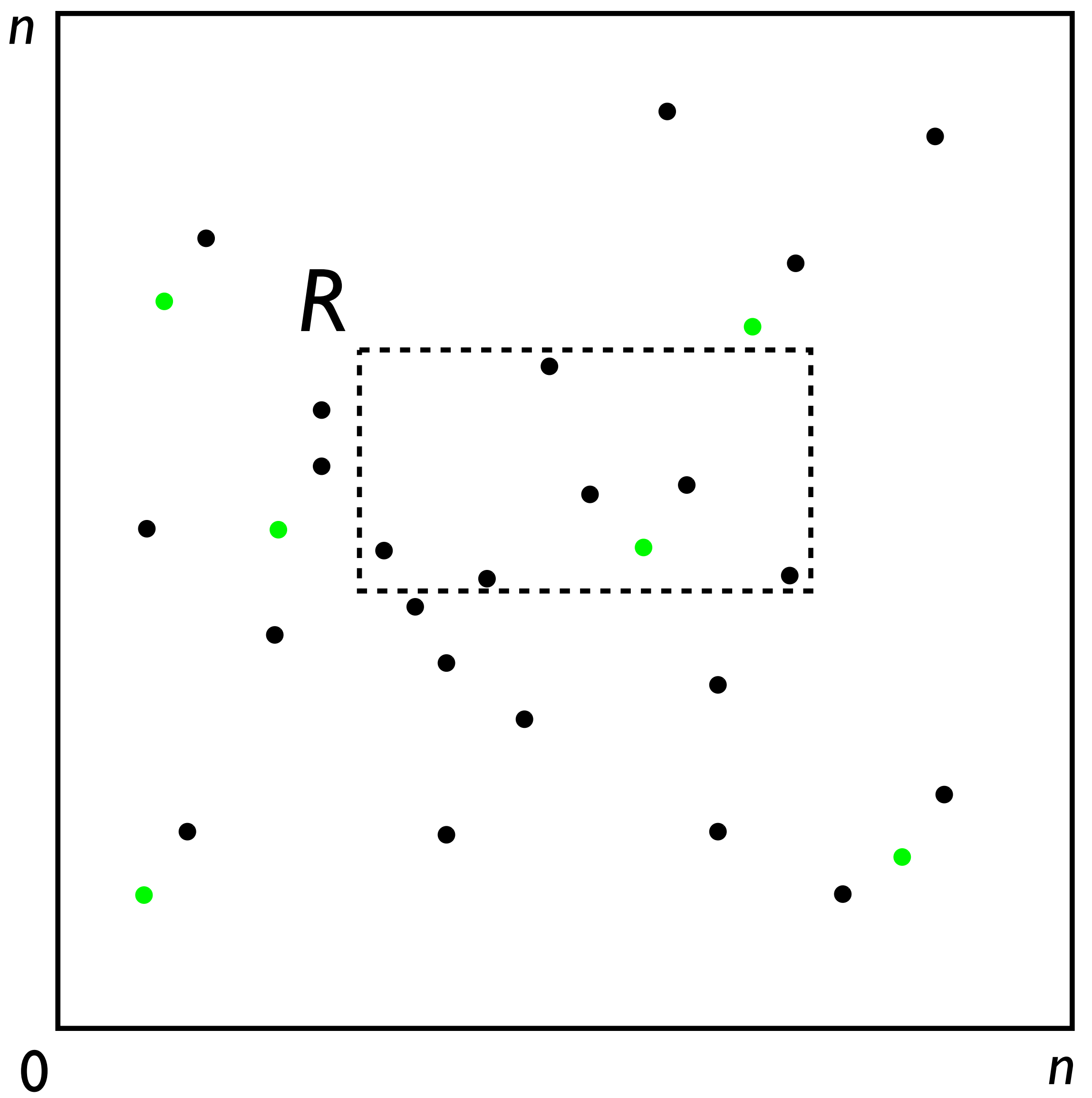

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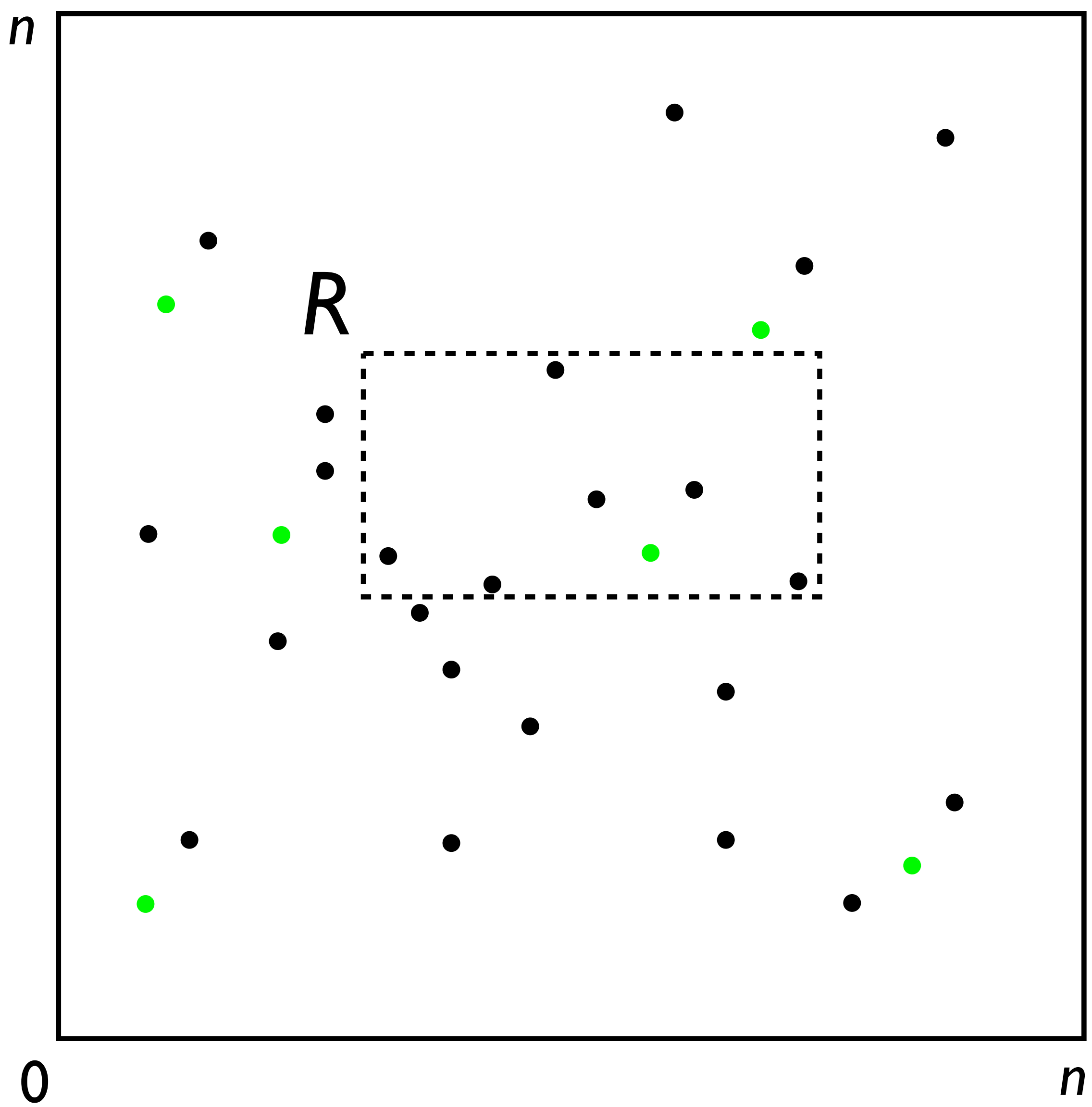

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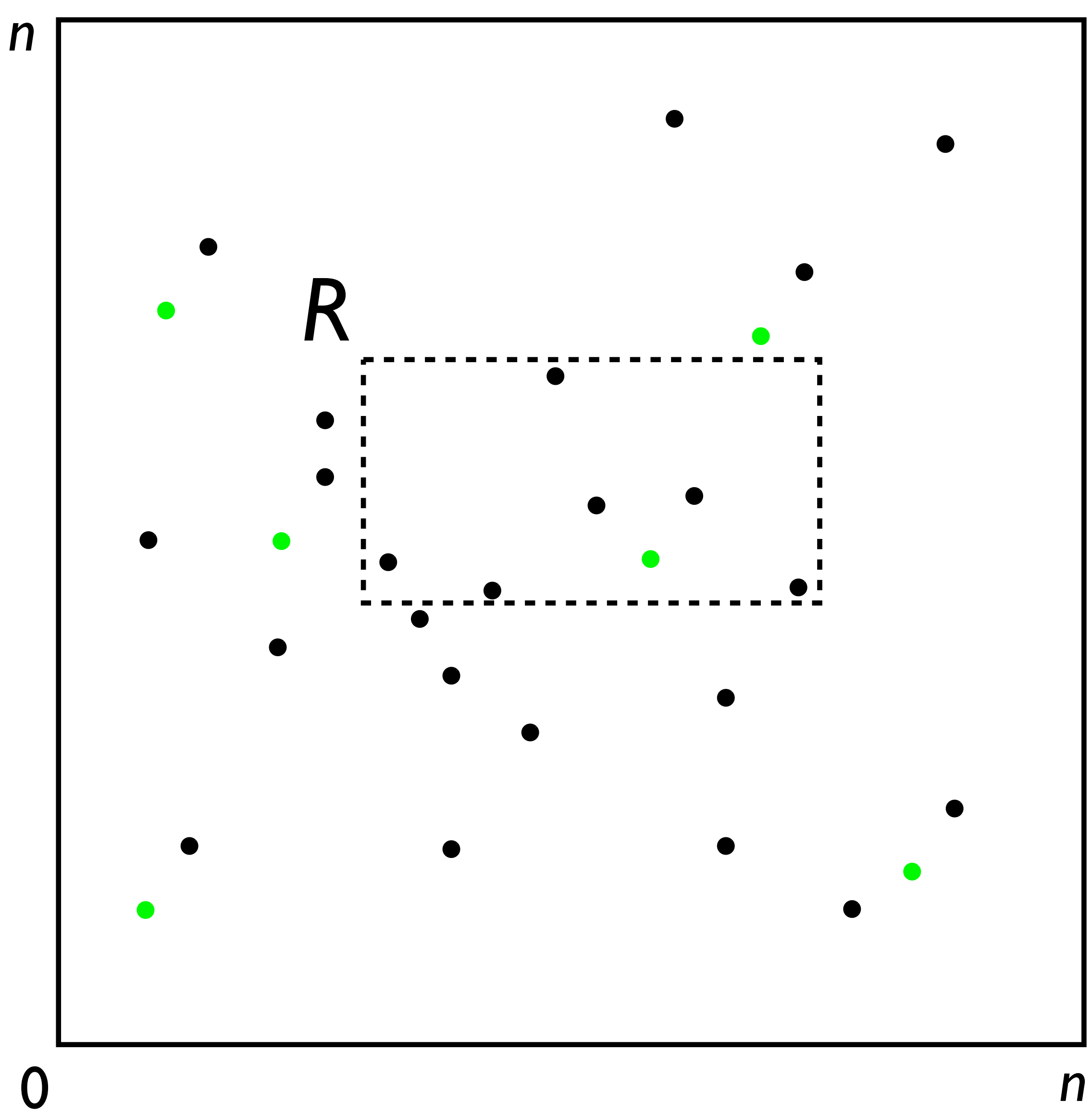

If $|R \cap P| \geq \varepsilon n$
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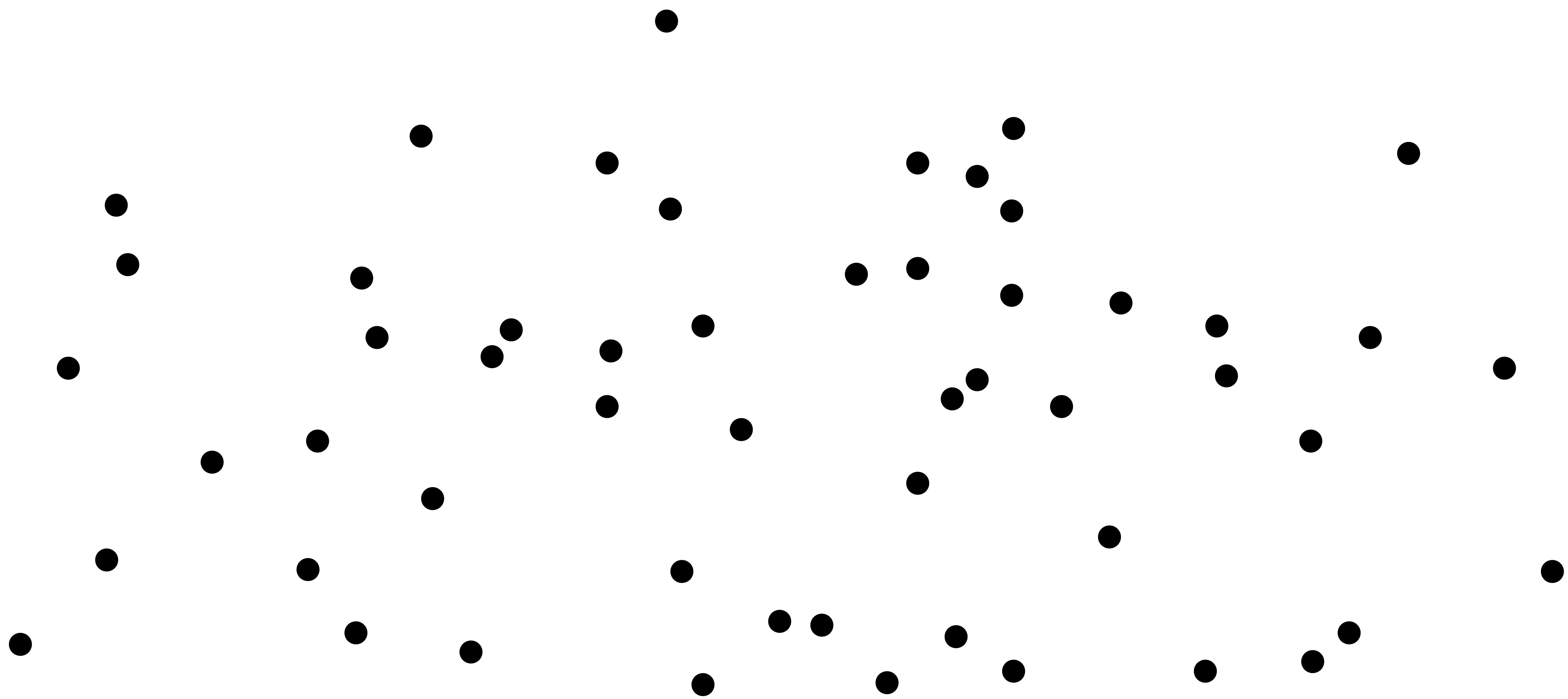
- $O(\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon})$ points.

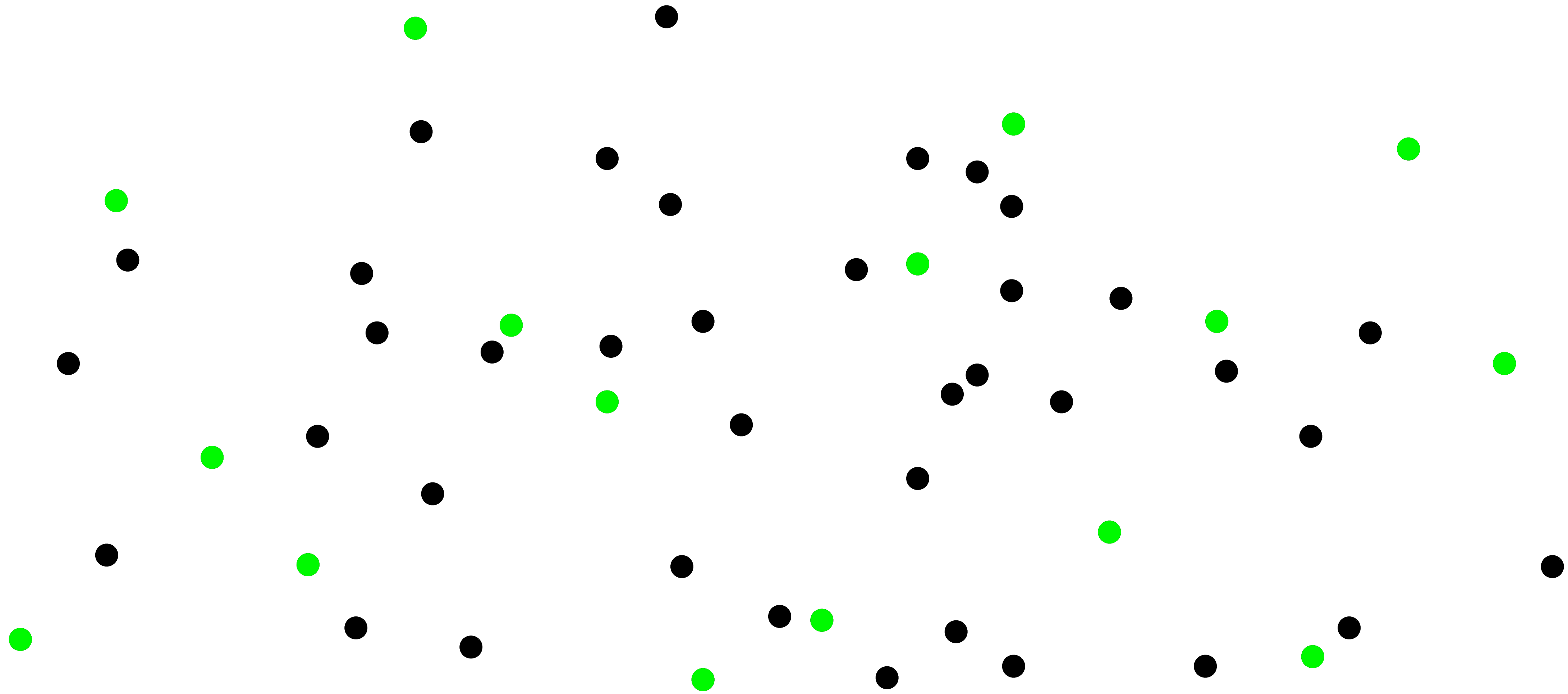
Upper Bound

Data Structure



Data Structure

- Take an ε-net of size $O(\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon})$.



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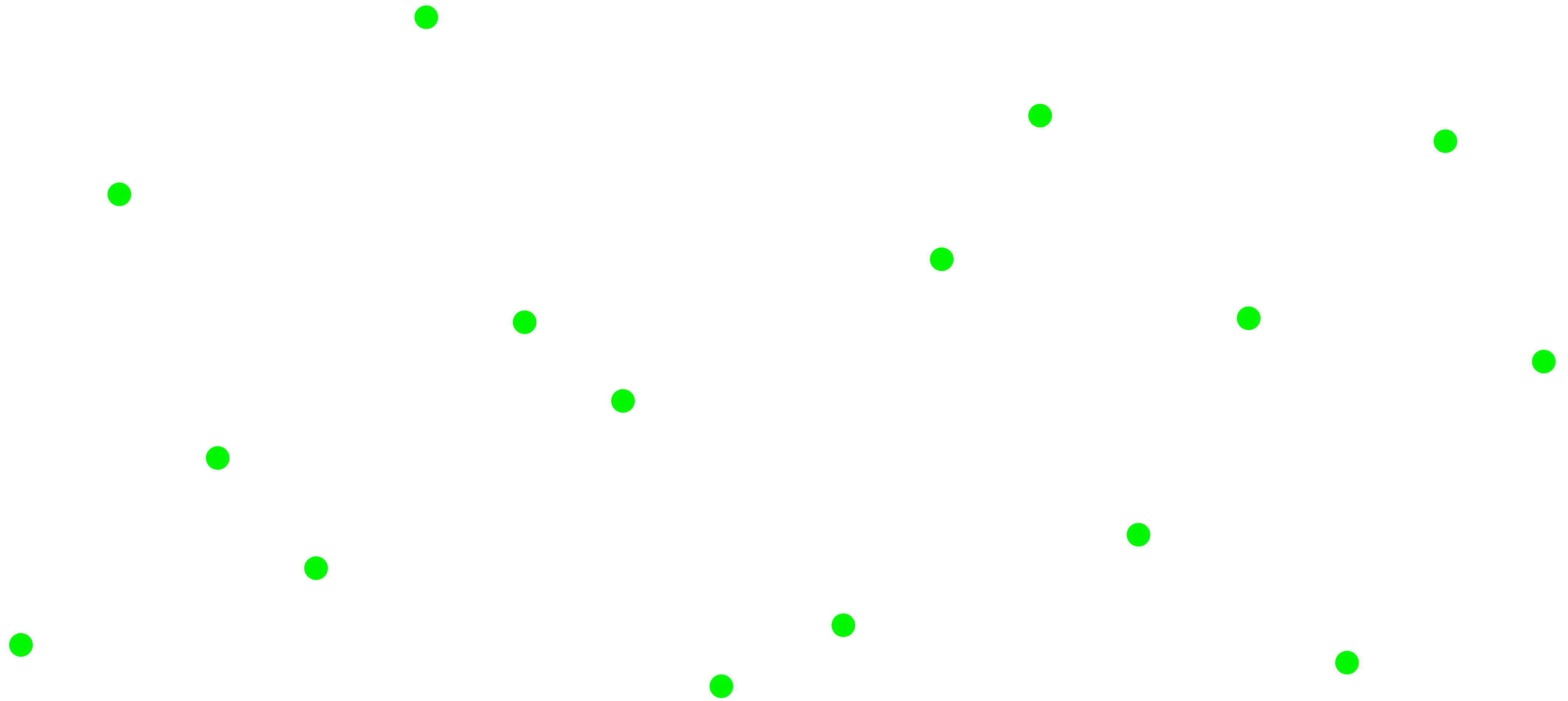

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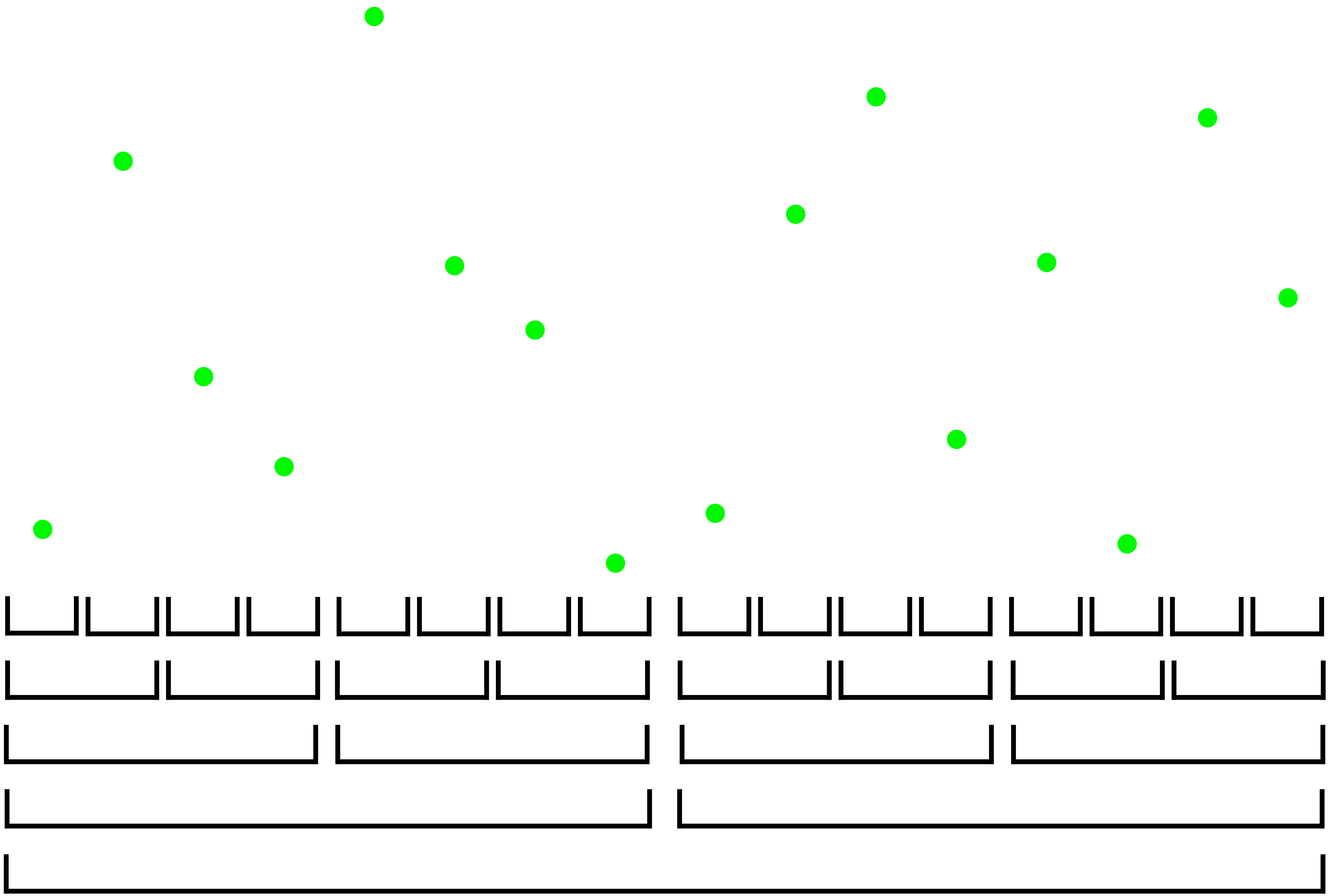

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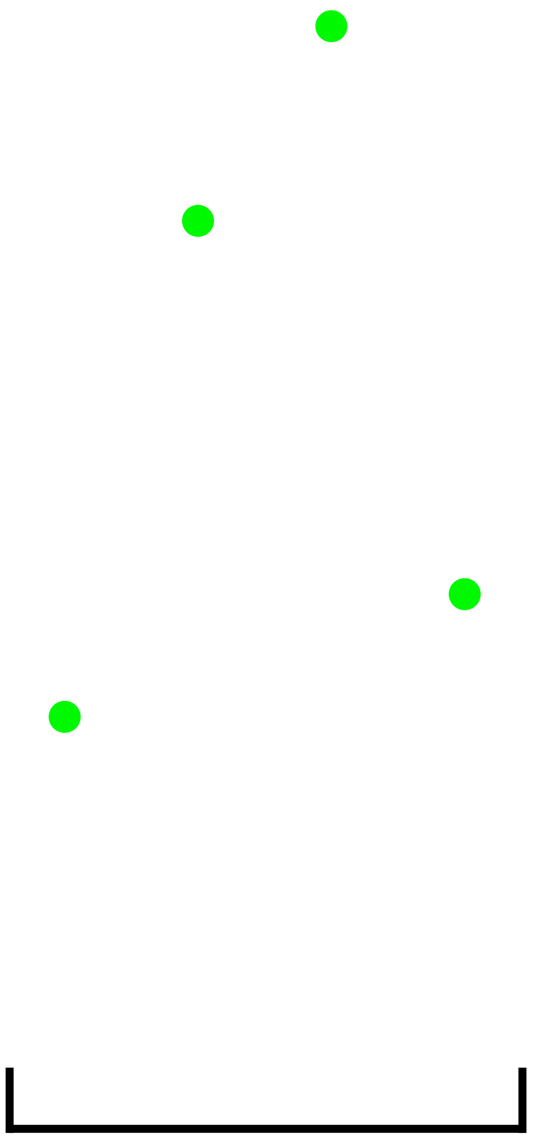

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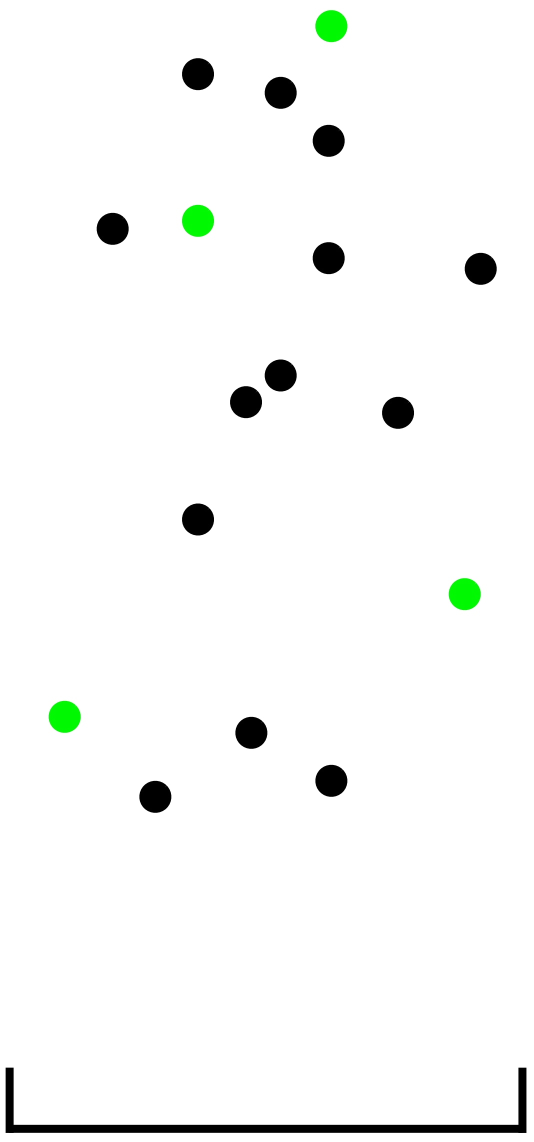

Data Structure

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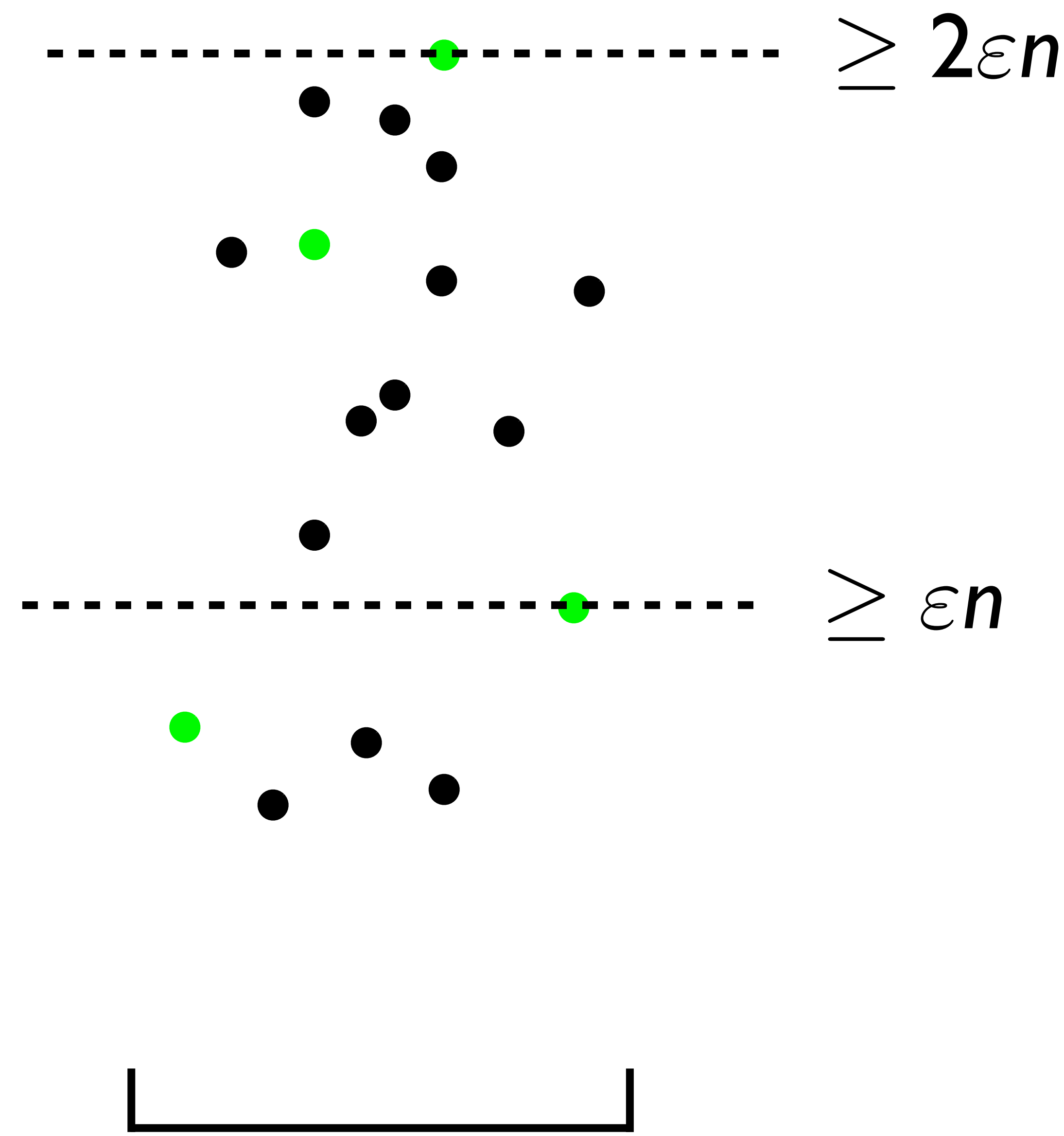

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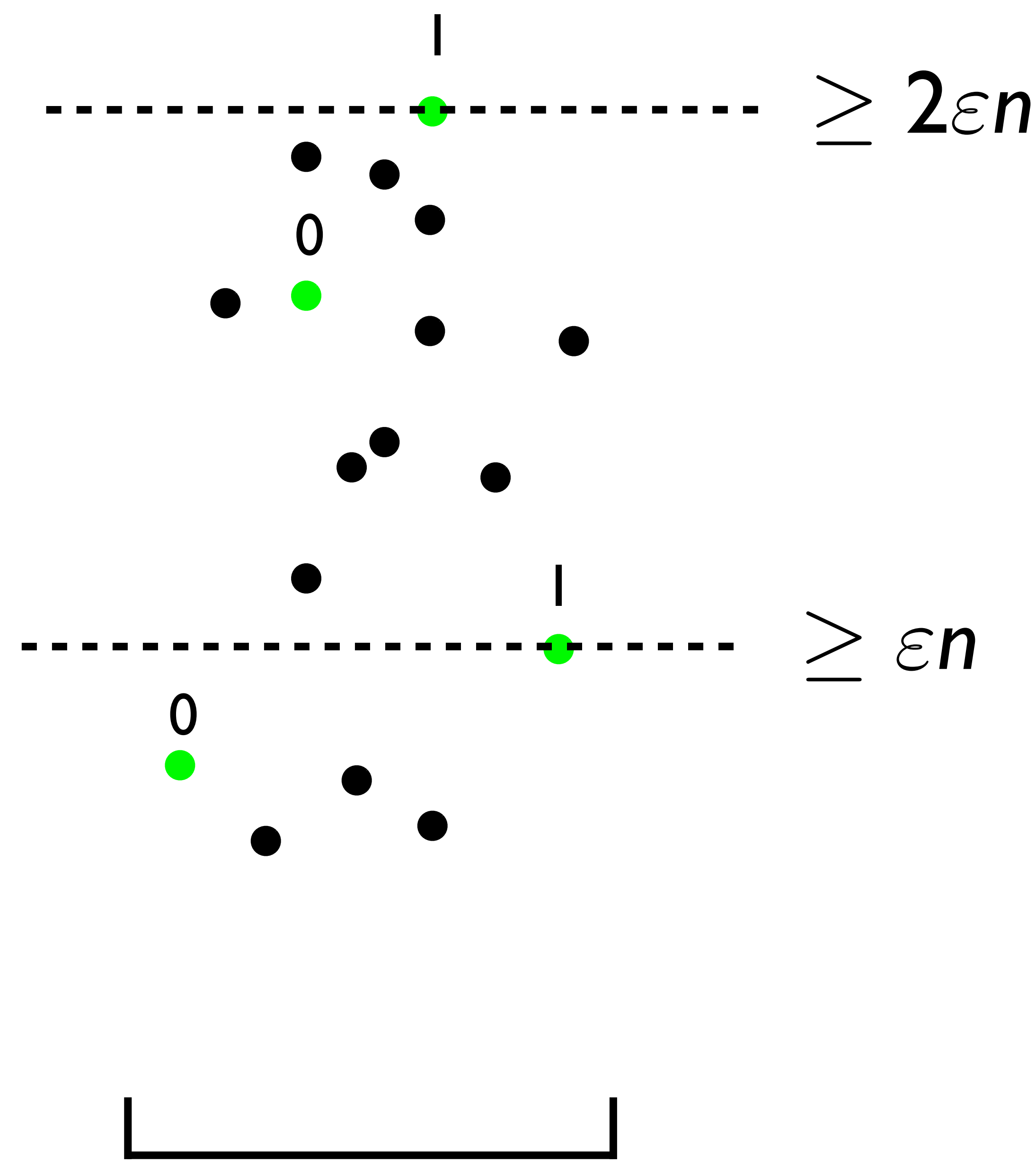

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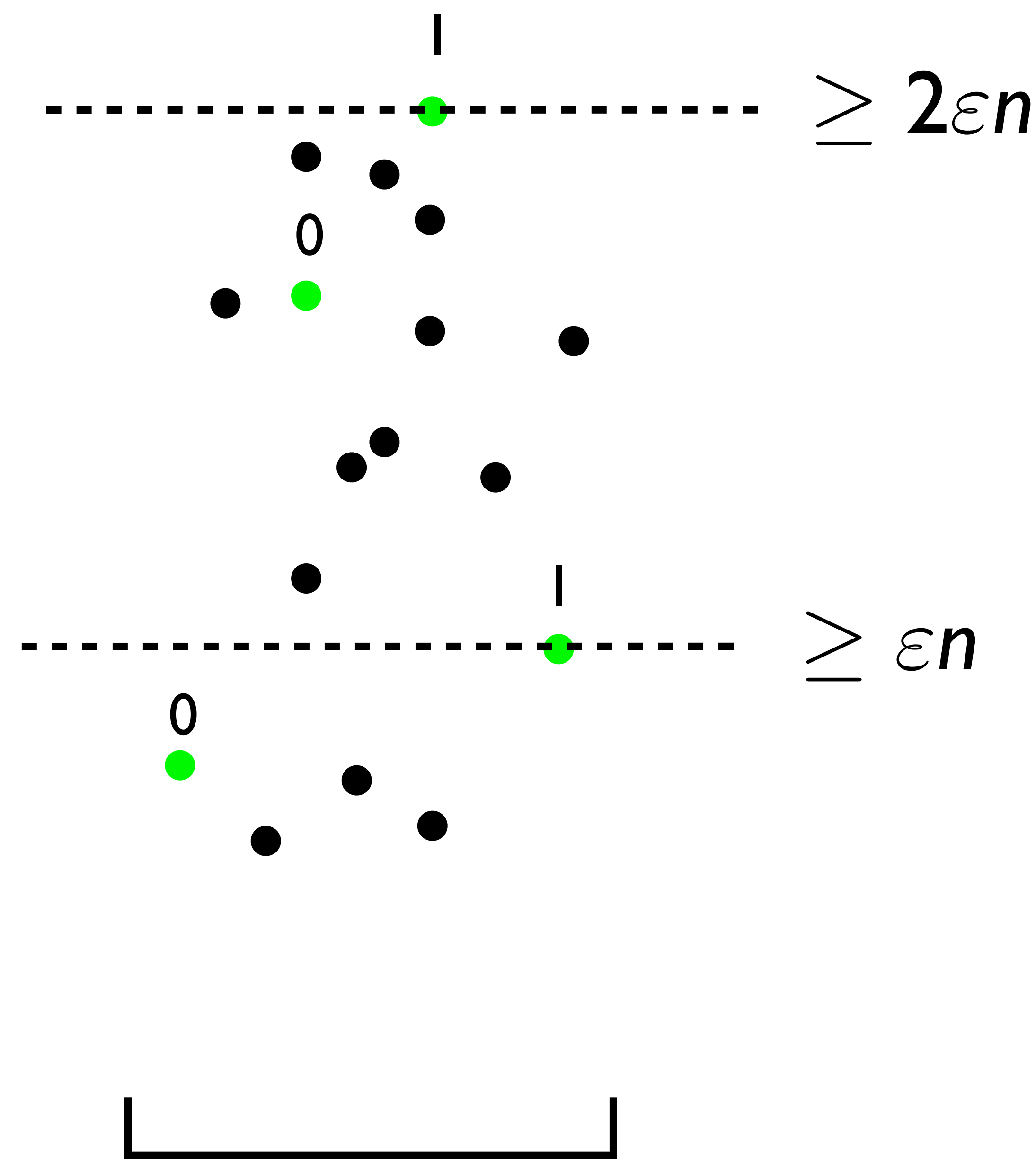
Each ε-net point in $O(\log \frac{1}{\varepsilon})$ canonical intervals.



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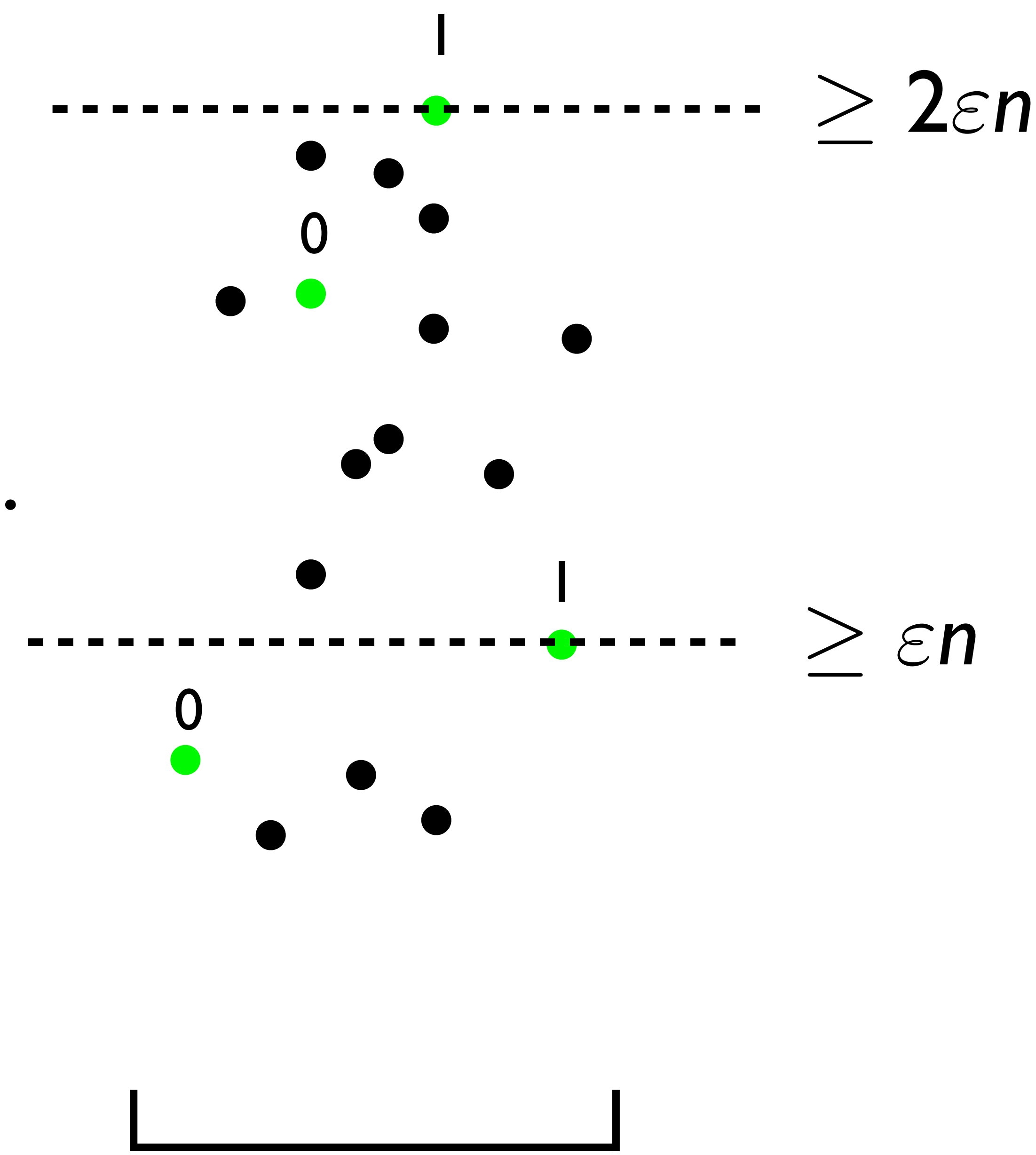

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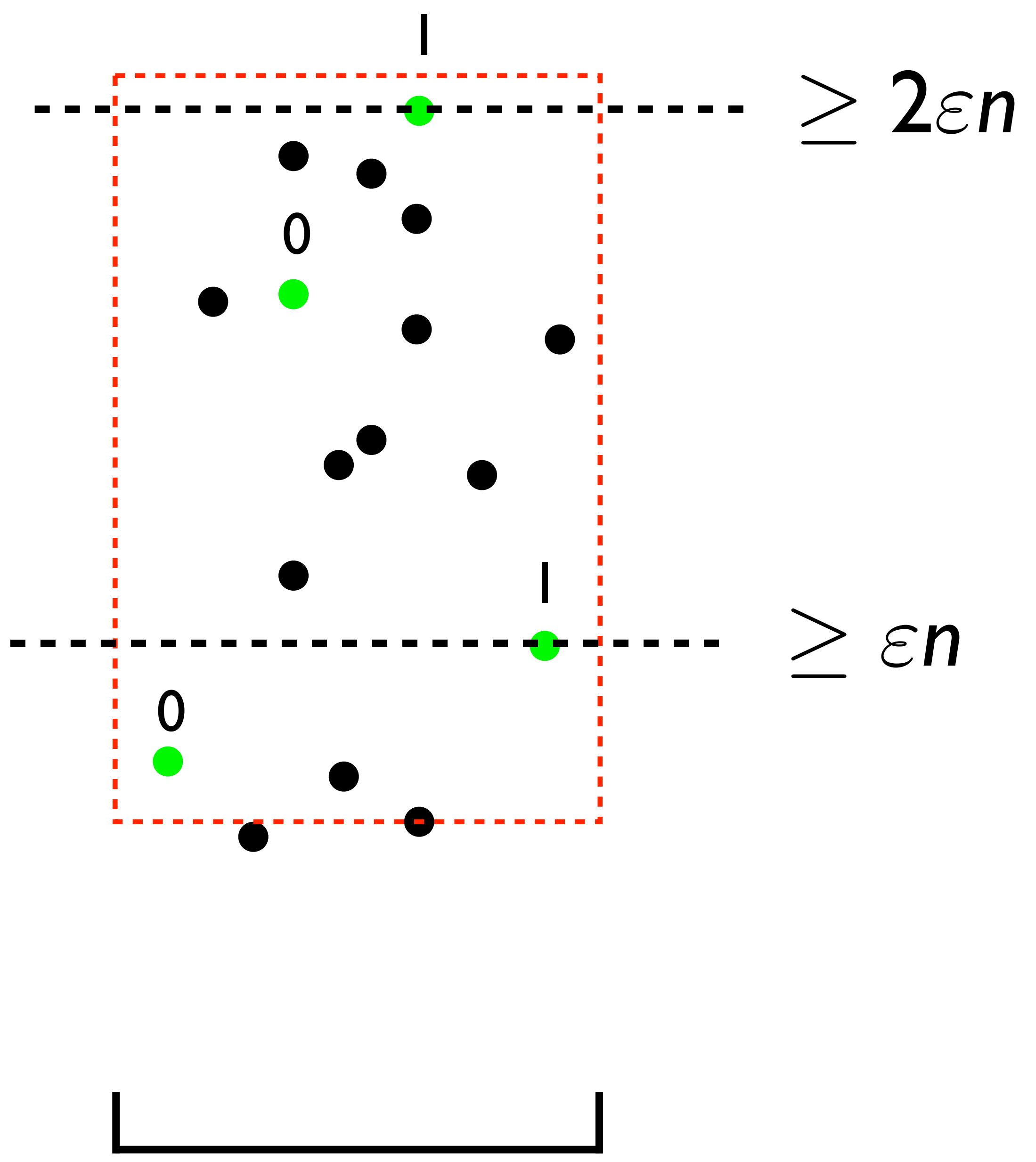

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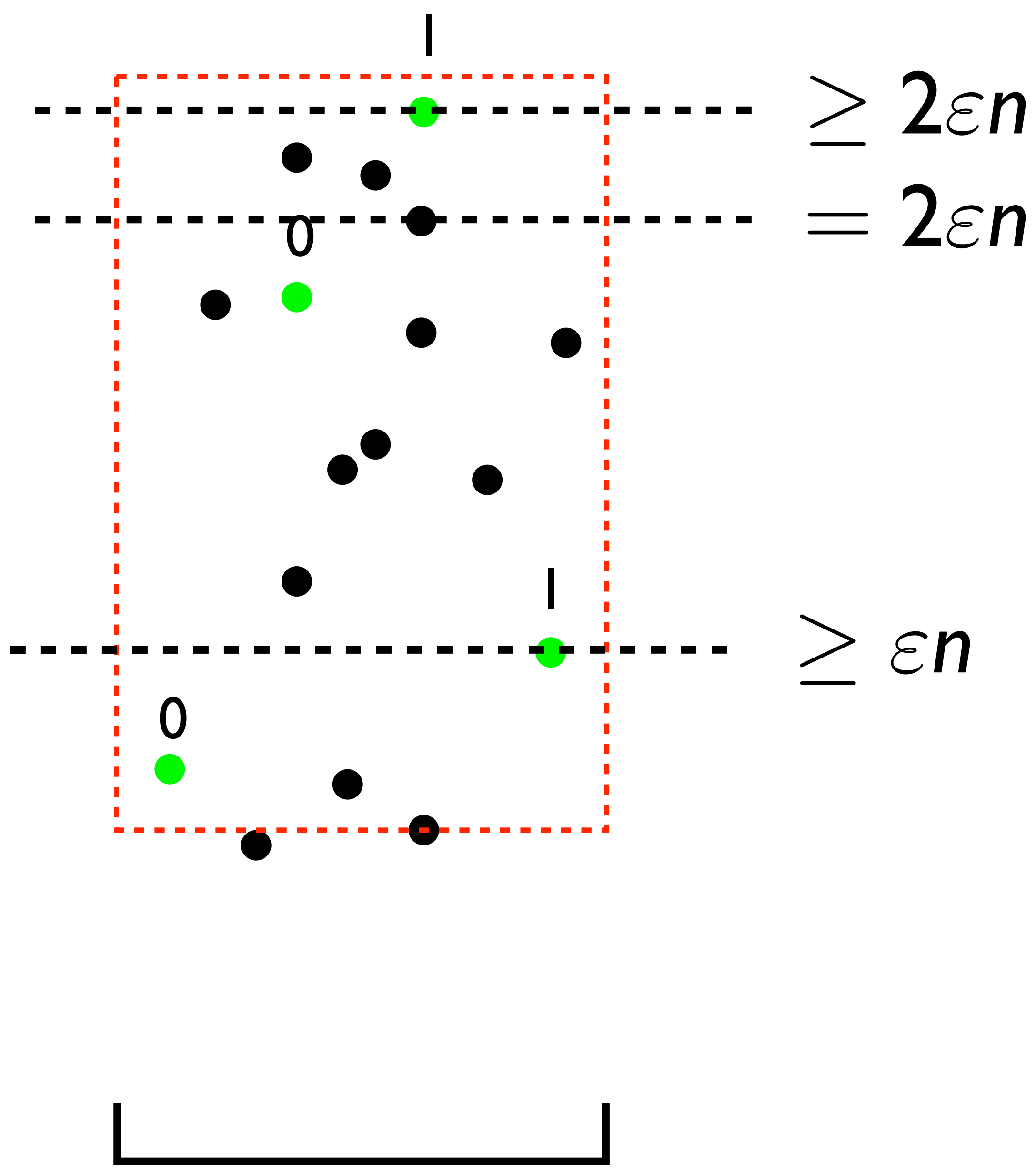

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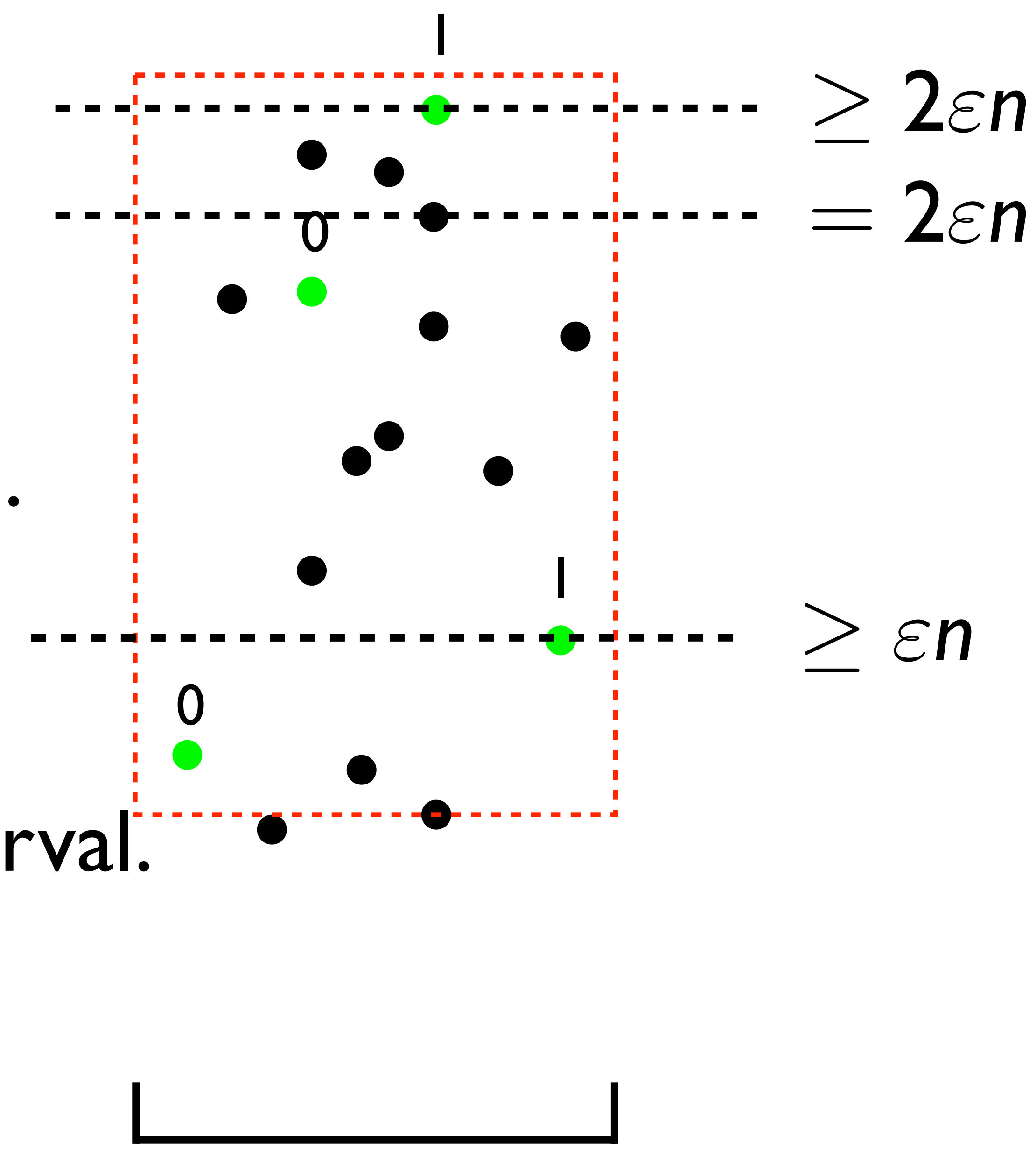

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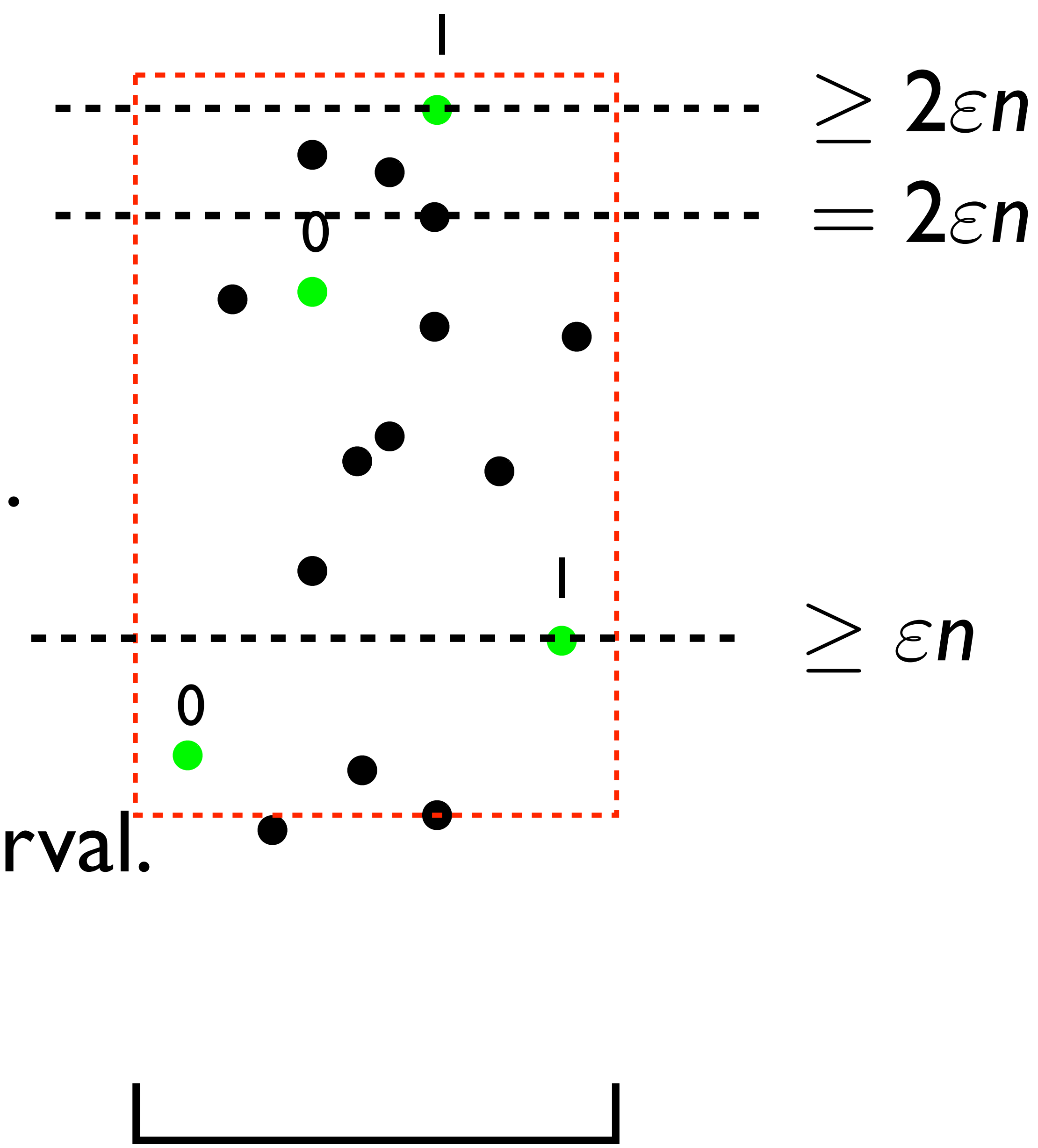

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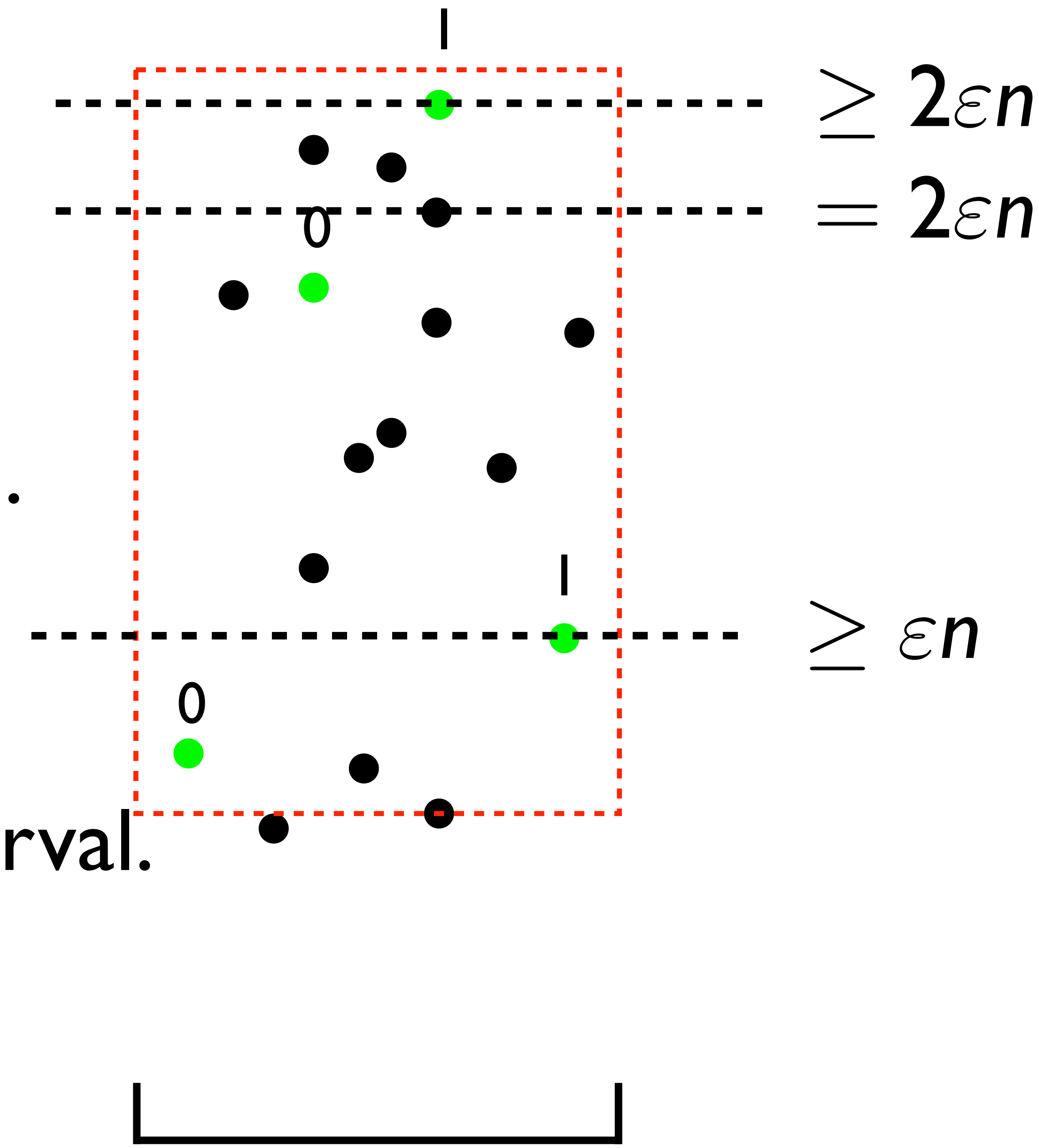

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Total error: $\varepsilon n \log \frac{1}{\varepsilon}$.

Setting $\varepsilon' = \frac{\varepsilon}{\log \frac{1}{\varepsilon}} \Rightarrow O(\frac{1}{\varepsilon'} \log\log \frac{1}{\varepsilon'} \log \frac{1}{\varepsilon'} \log n)$ bits.

Lower Bound

Data Structure to CD

- Lower bound: $\Omega(n \log n)$ bits needed for error $\log n$.

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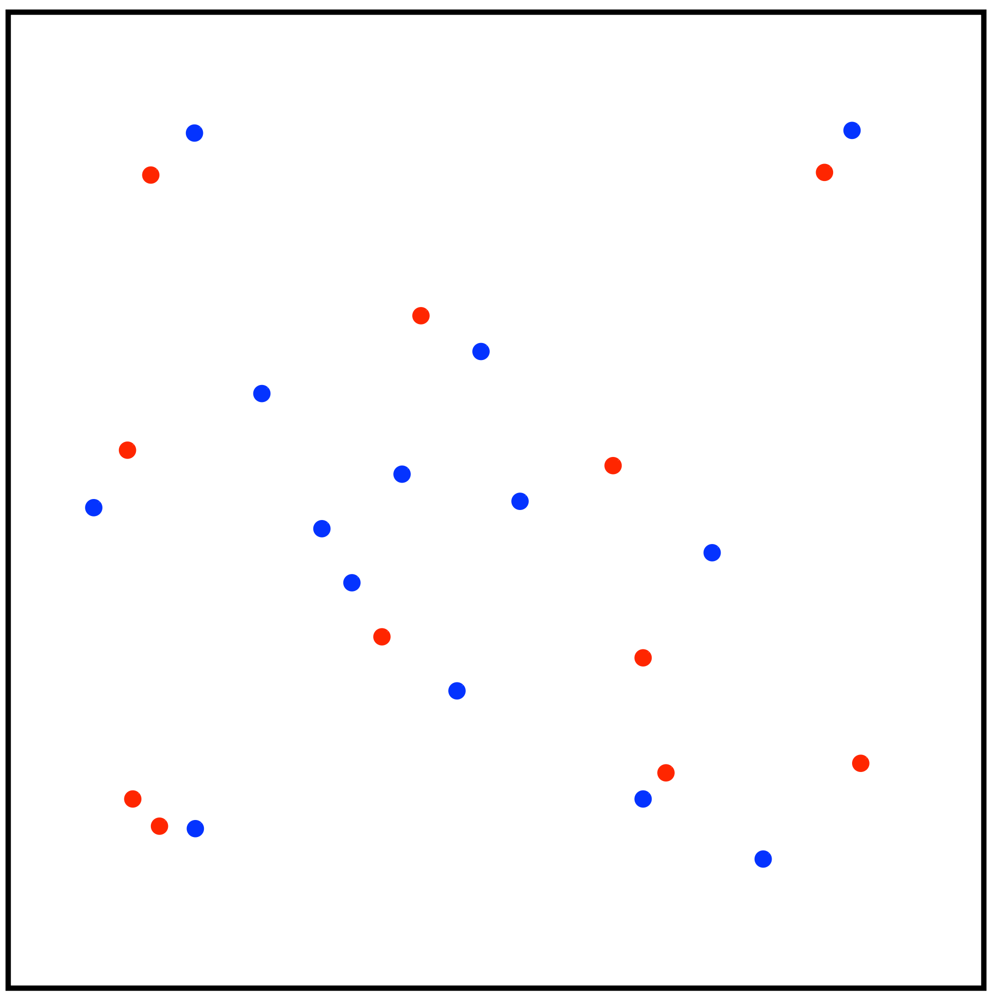

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$$\chi(p) = \begin{cases} 1 & \text{if } p \in P_1; \\ -1 & \text{if } p \in P_2. \end{cases}$$

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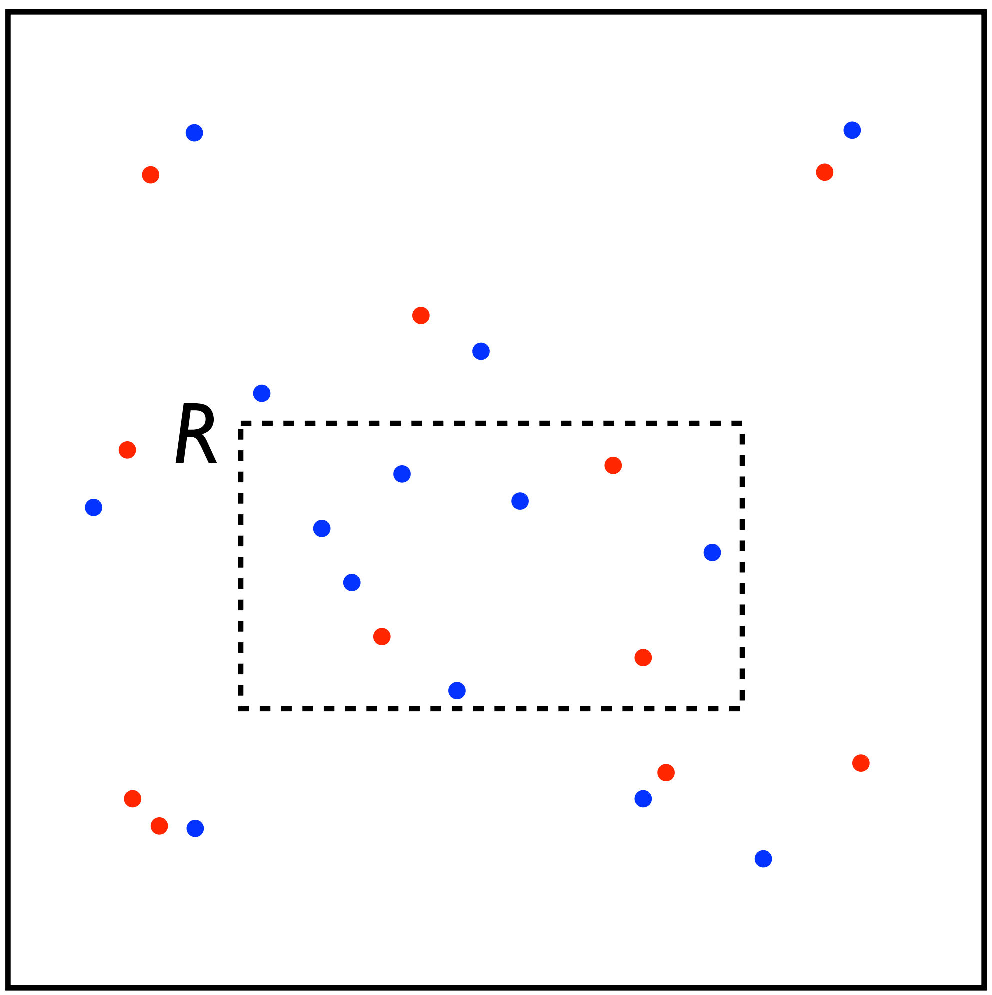

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$$||R \cap P_1| - |R \cap P_2|| \geq c \log n$$

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  - $O(2^n)$ point sets.

Binary Nets

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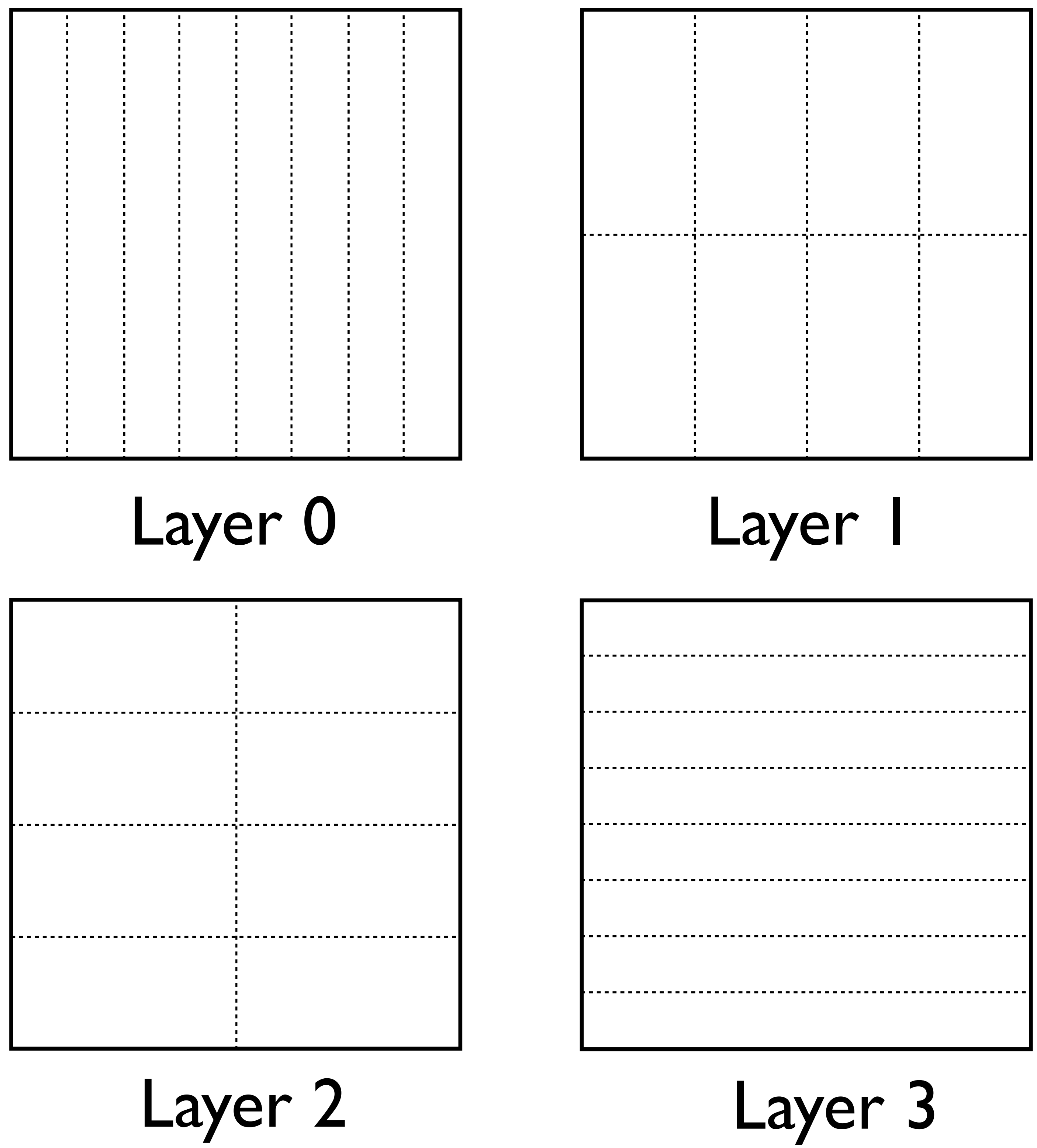
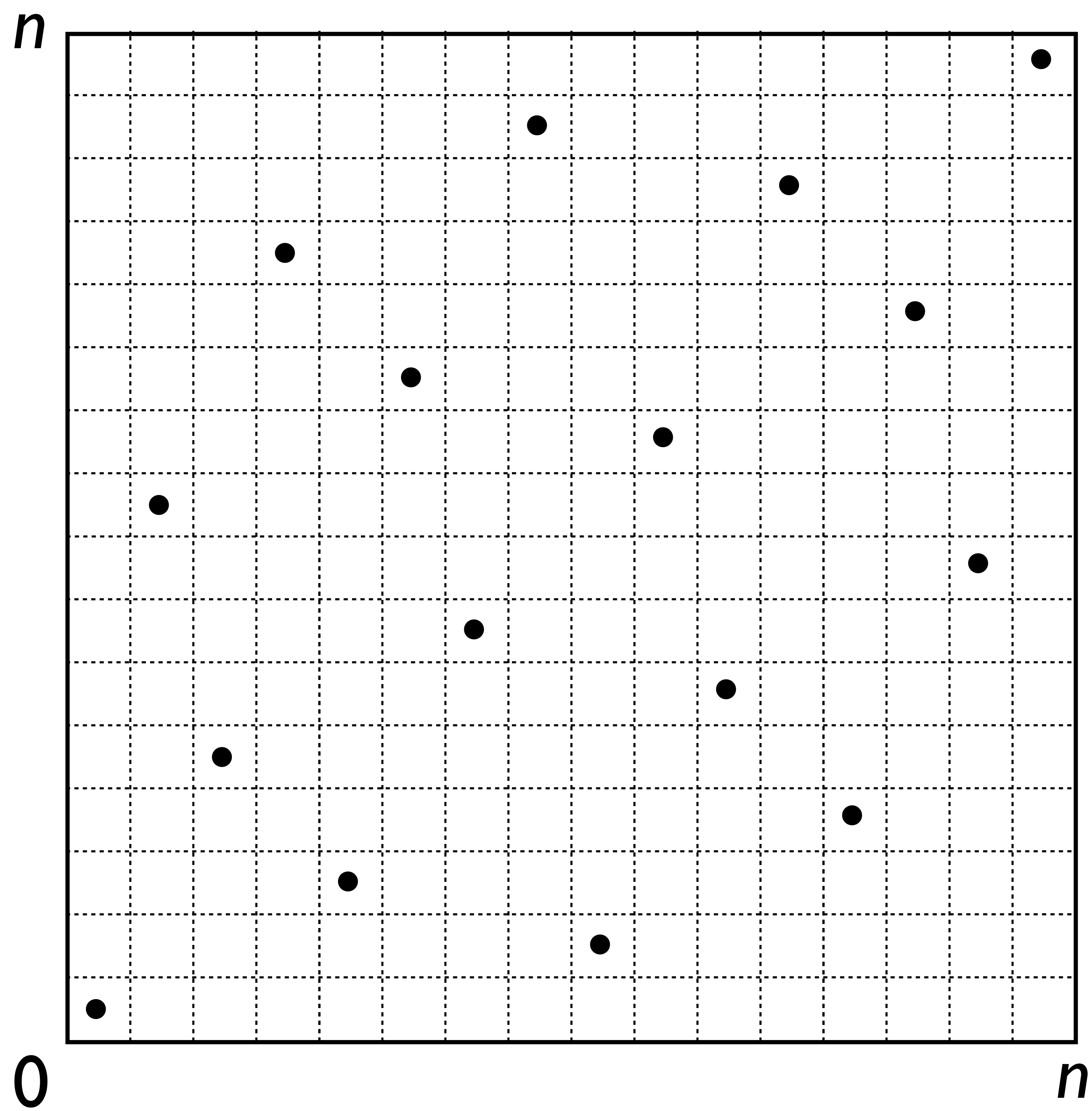
Canonical Cells



$n = 8$

Binary Nets



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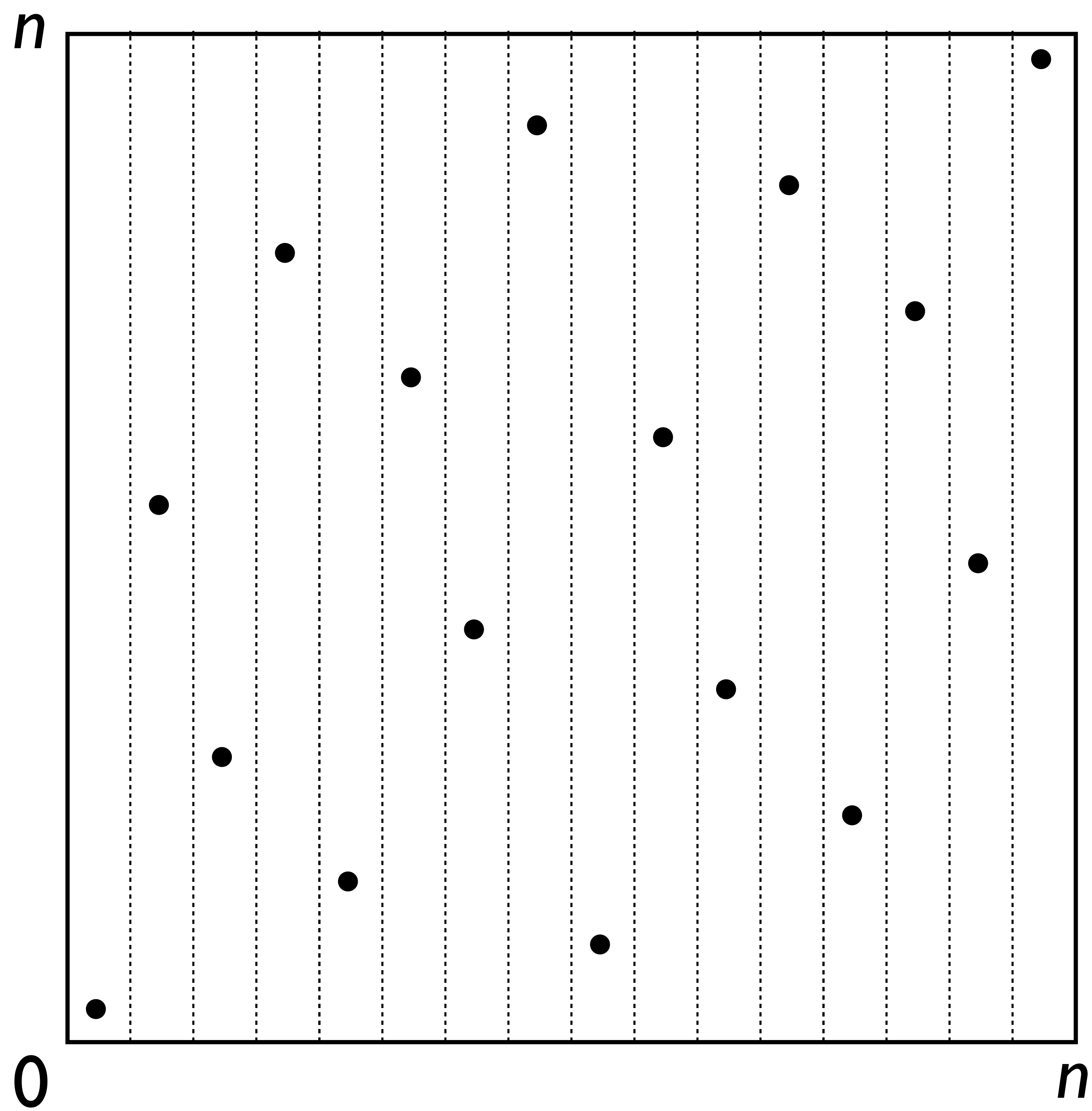
Binary Nets



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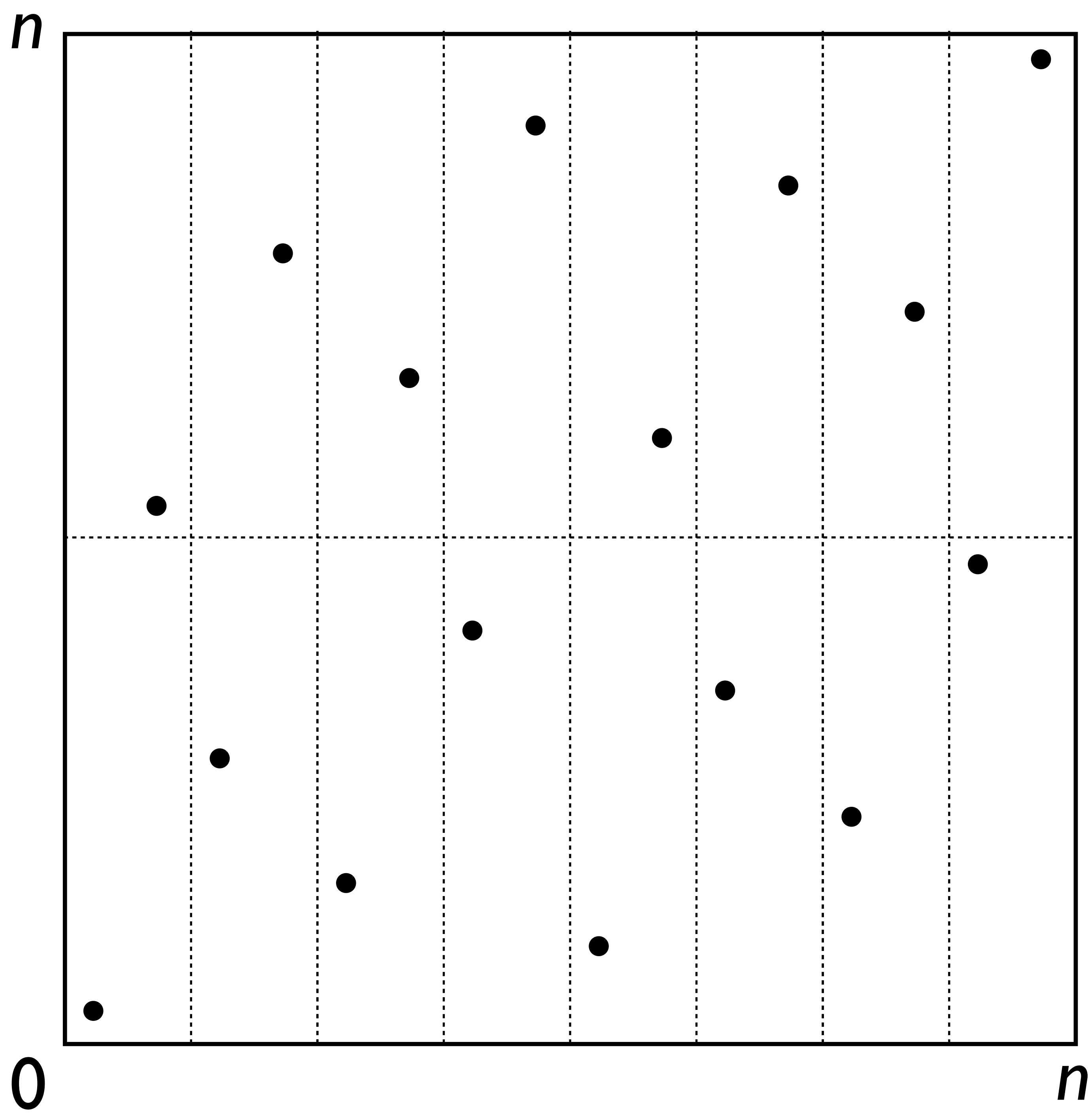

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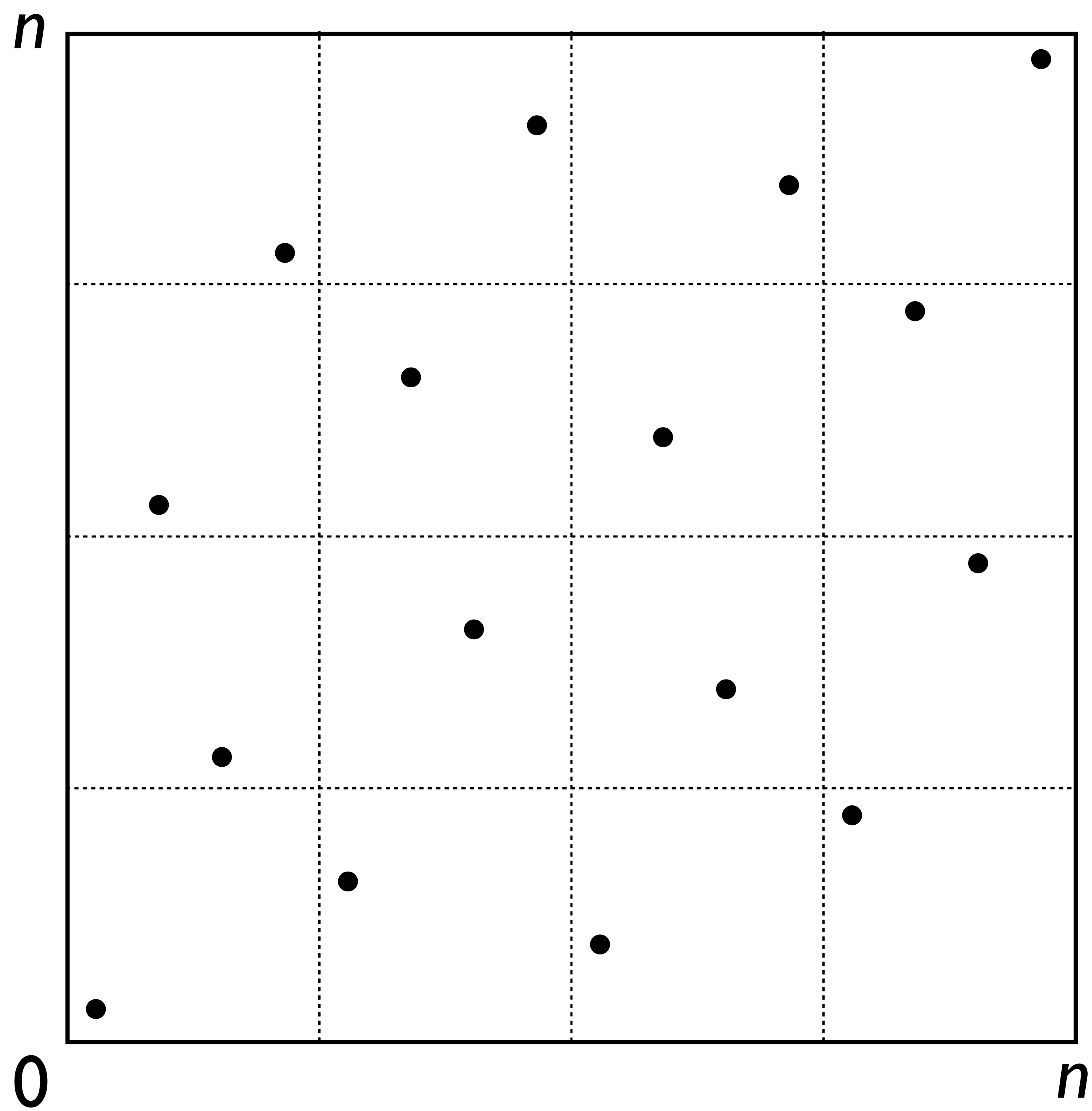

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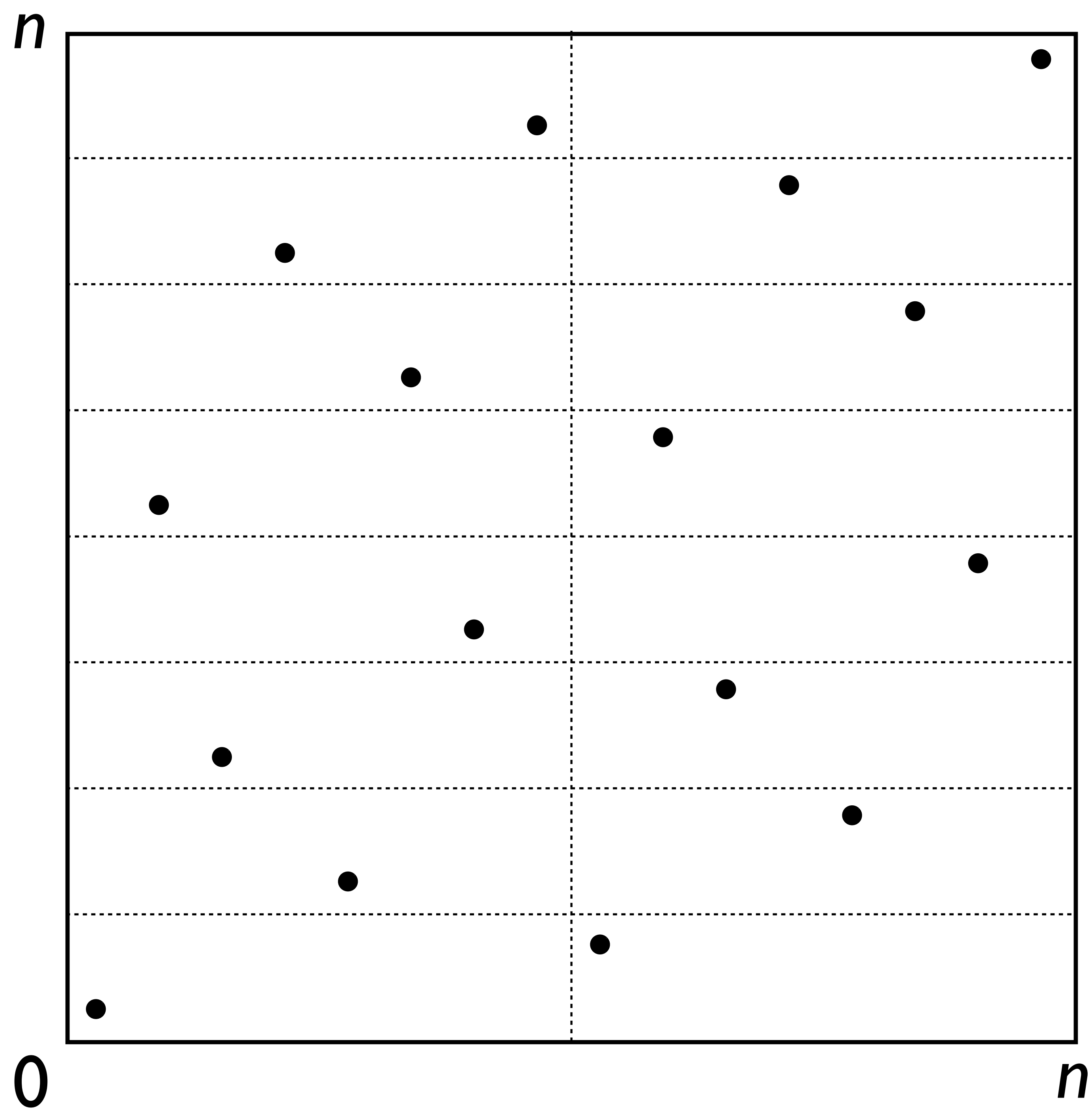

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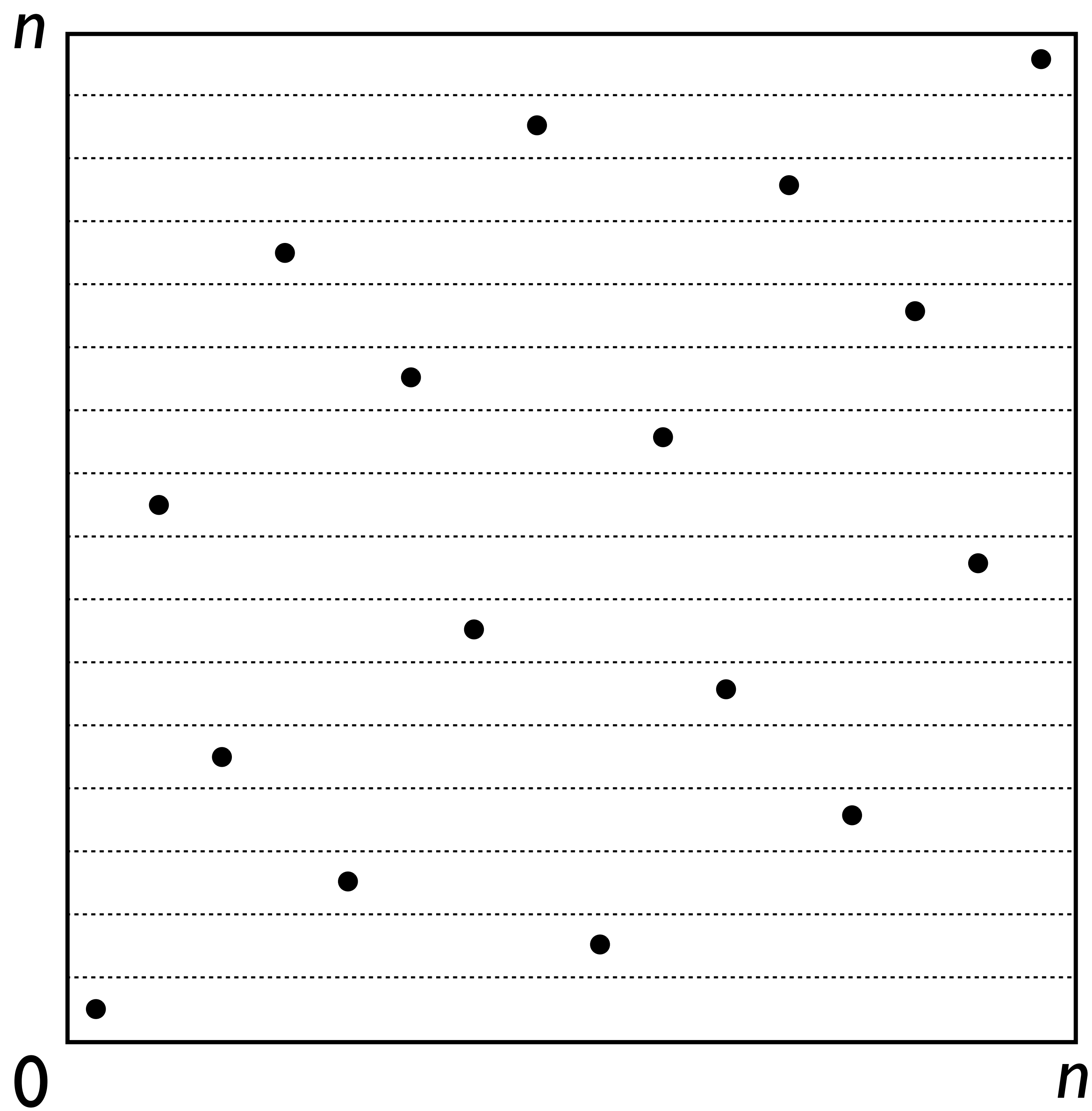

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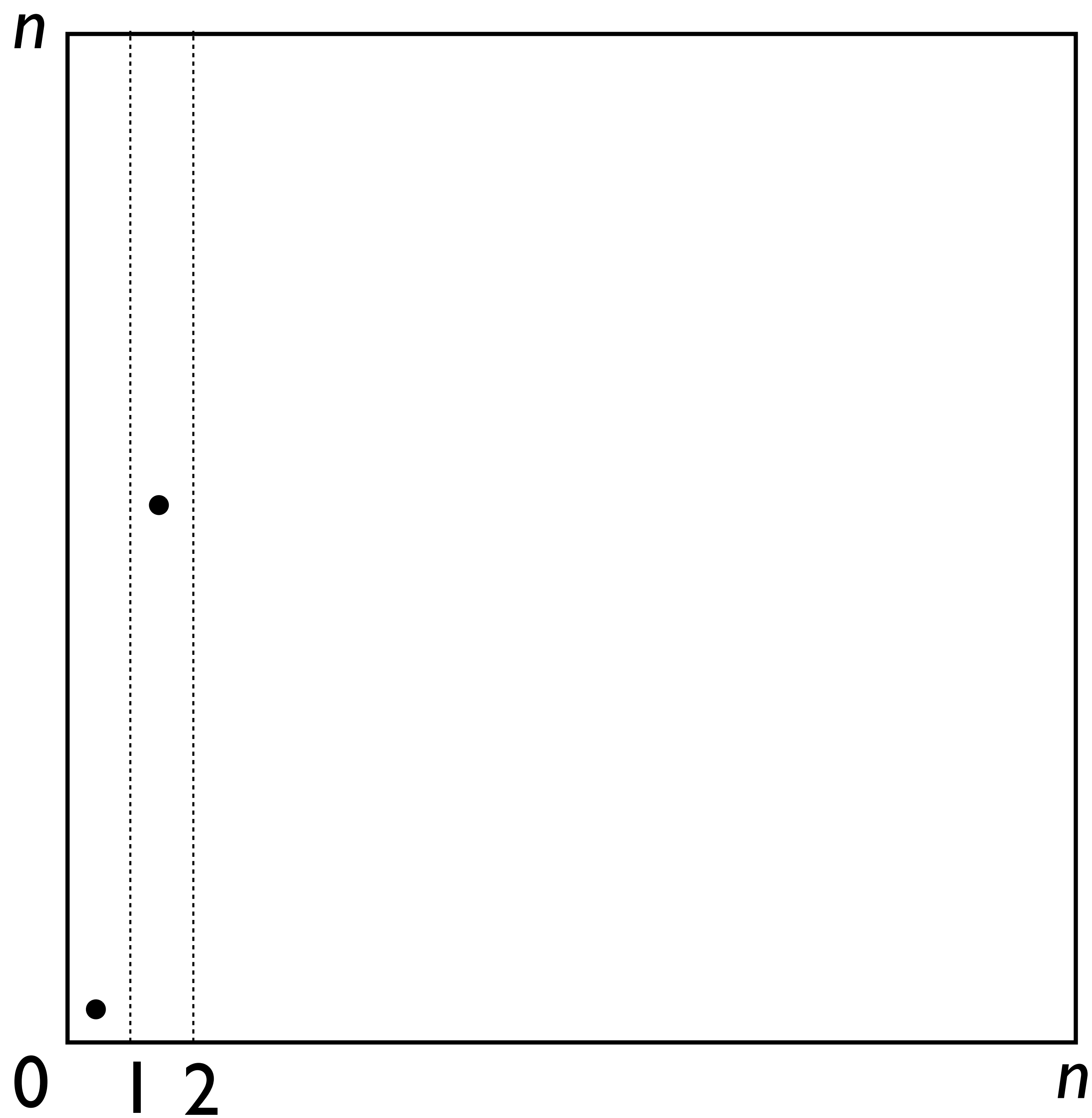
Number of Binary Nets



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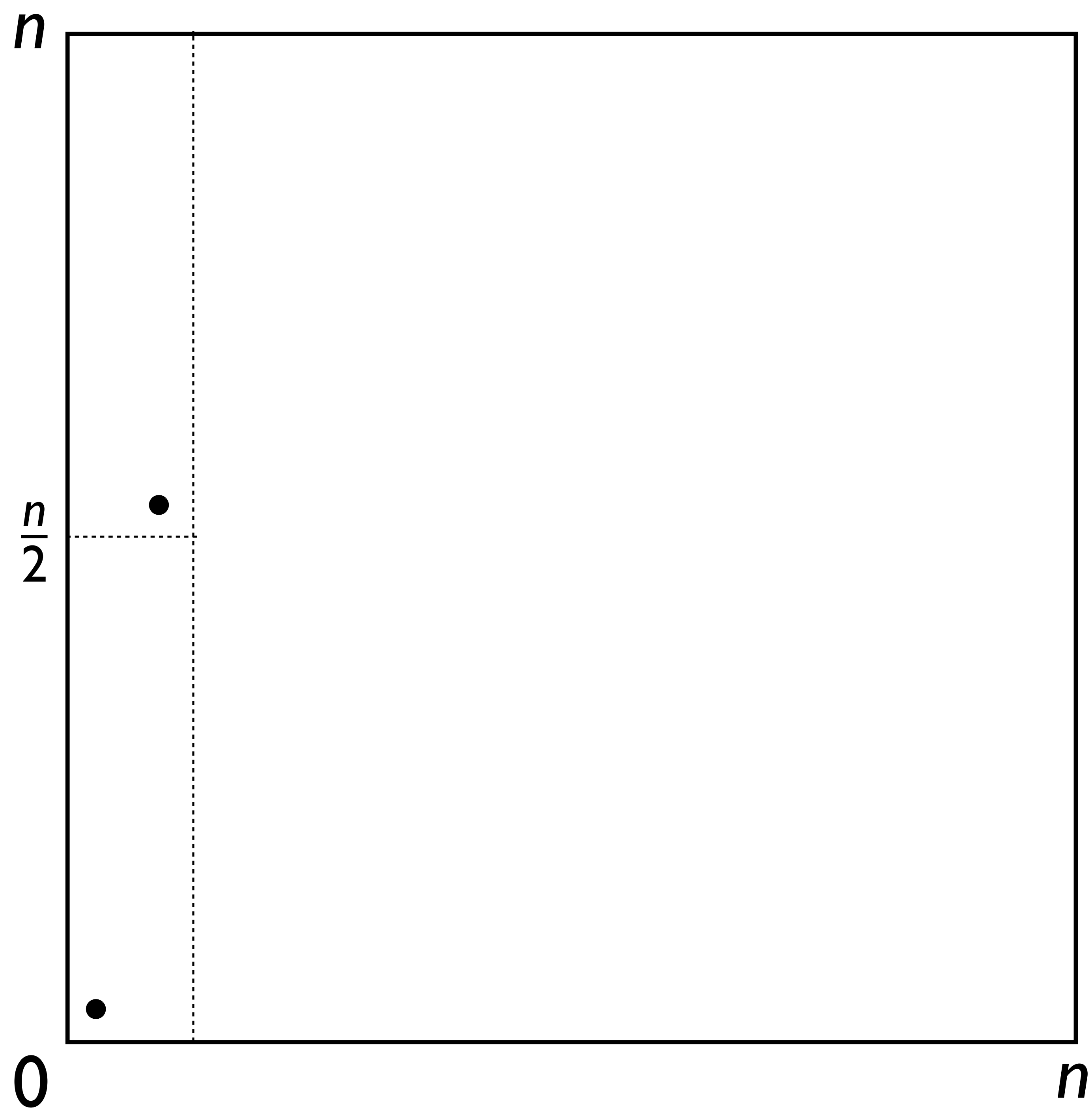

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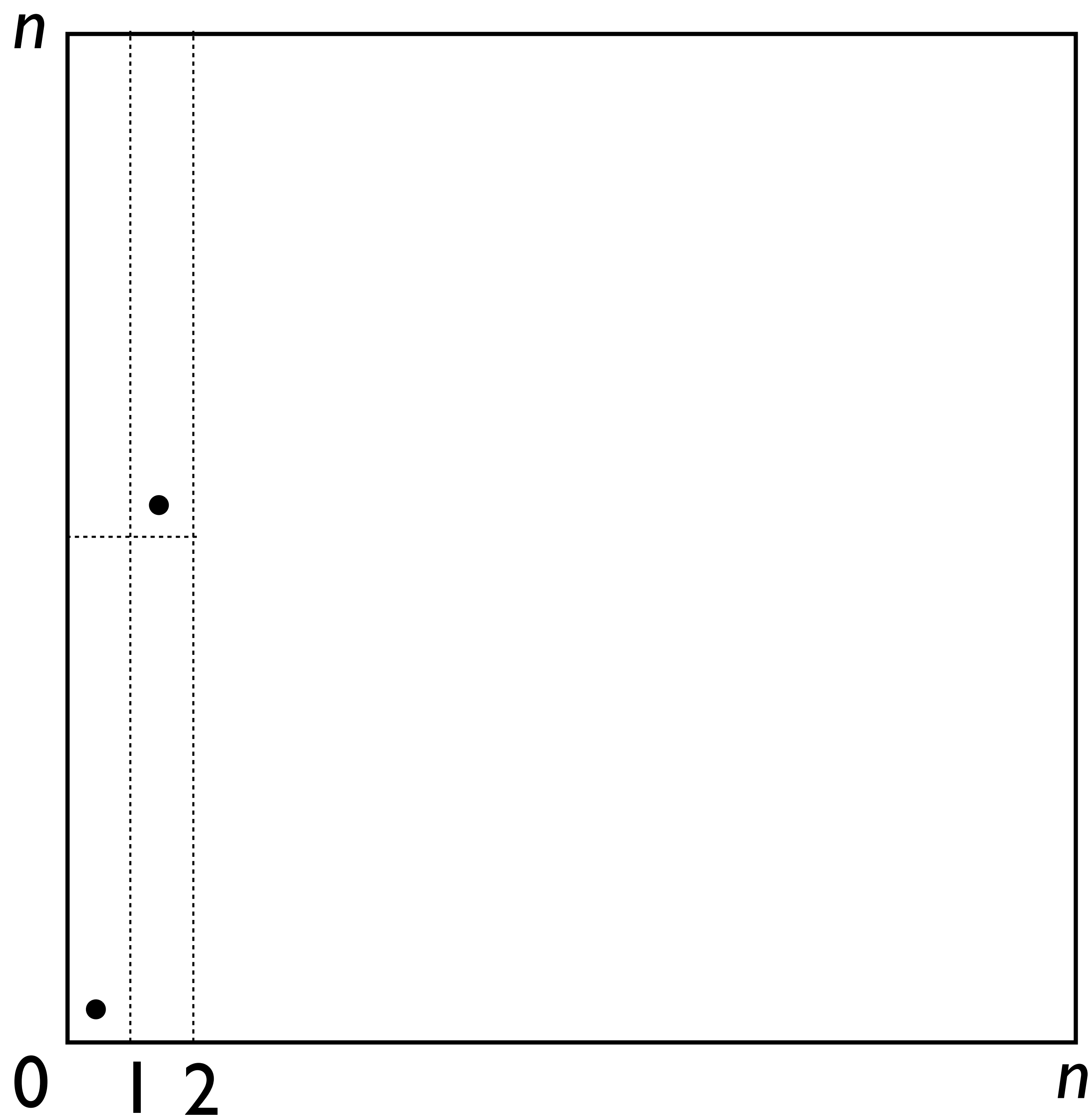

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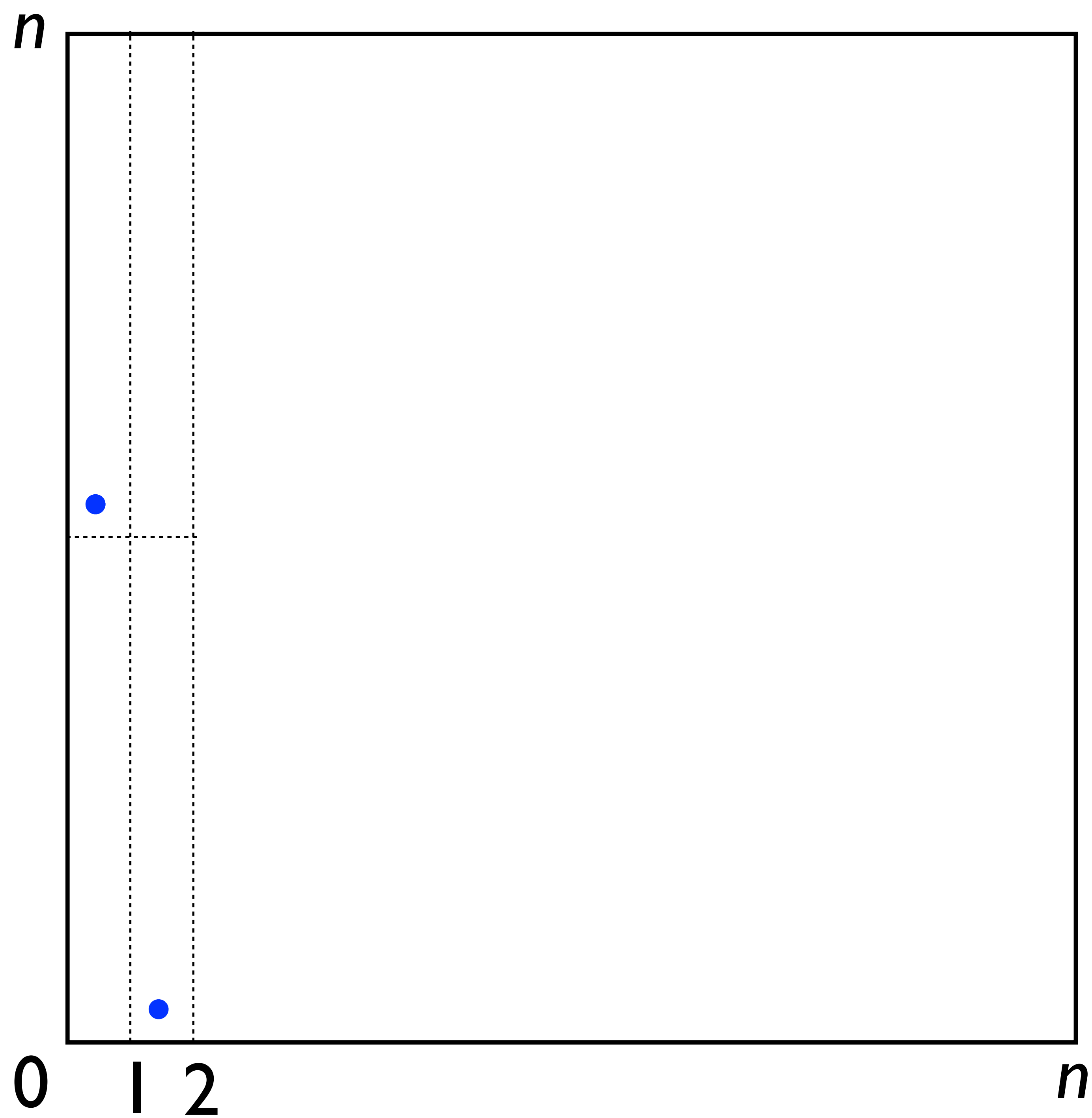

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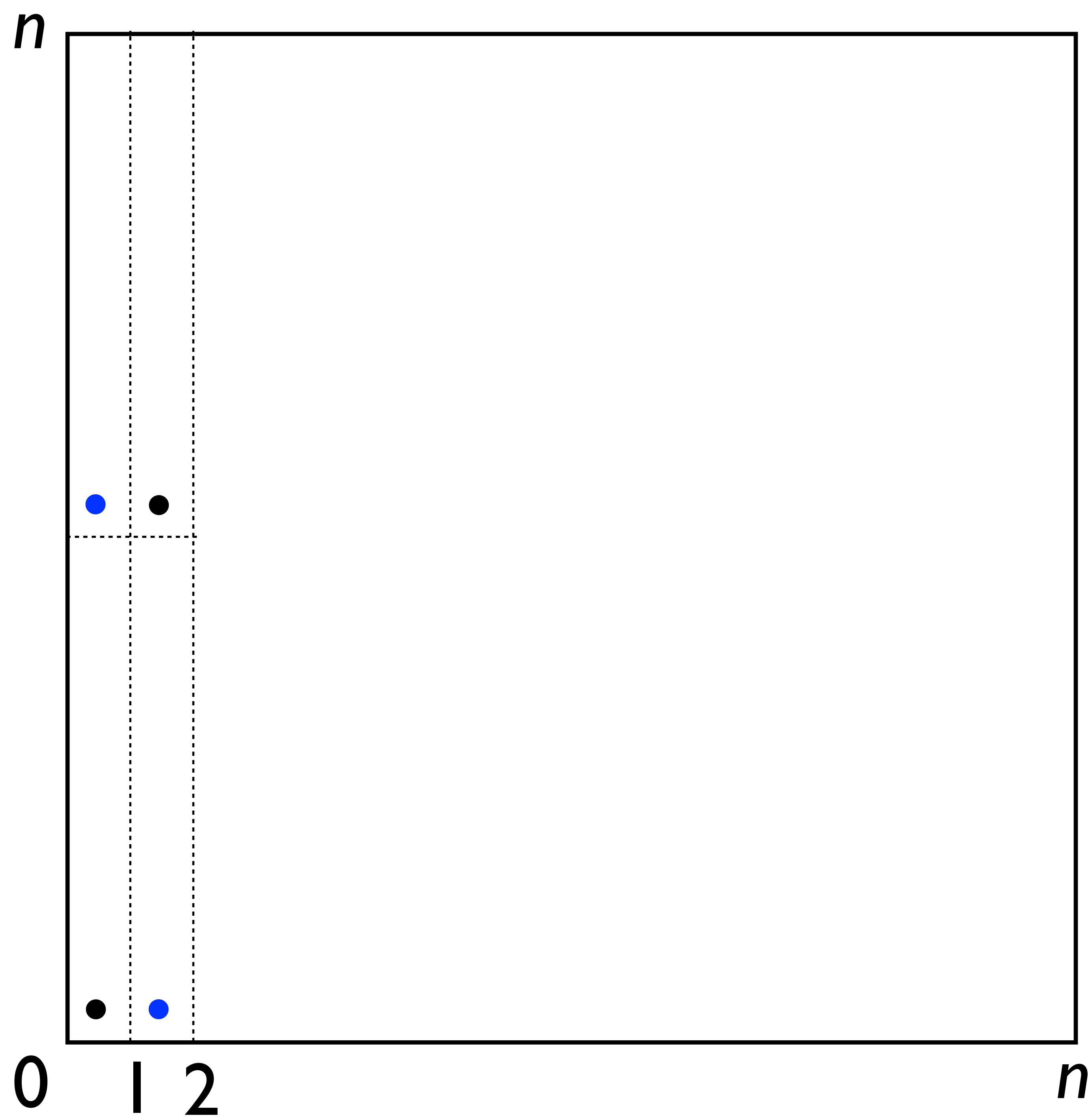

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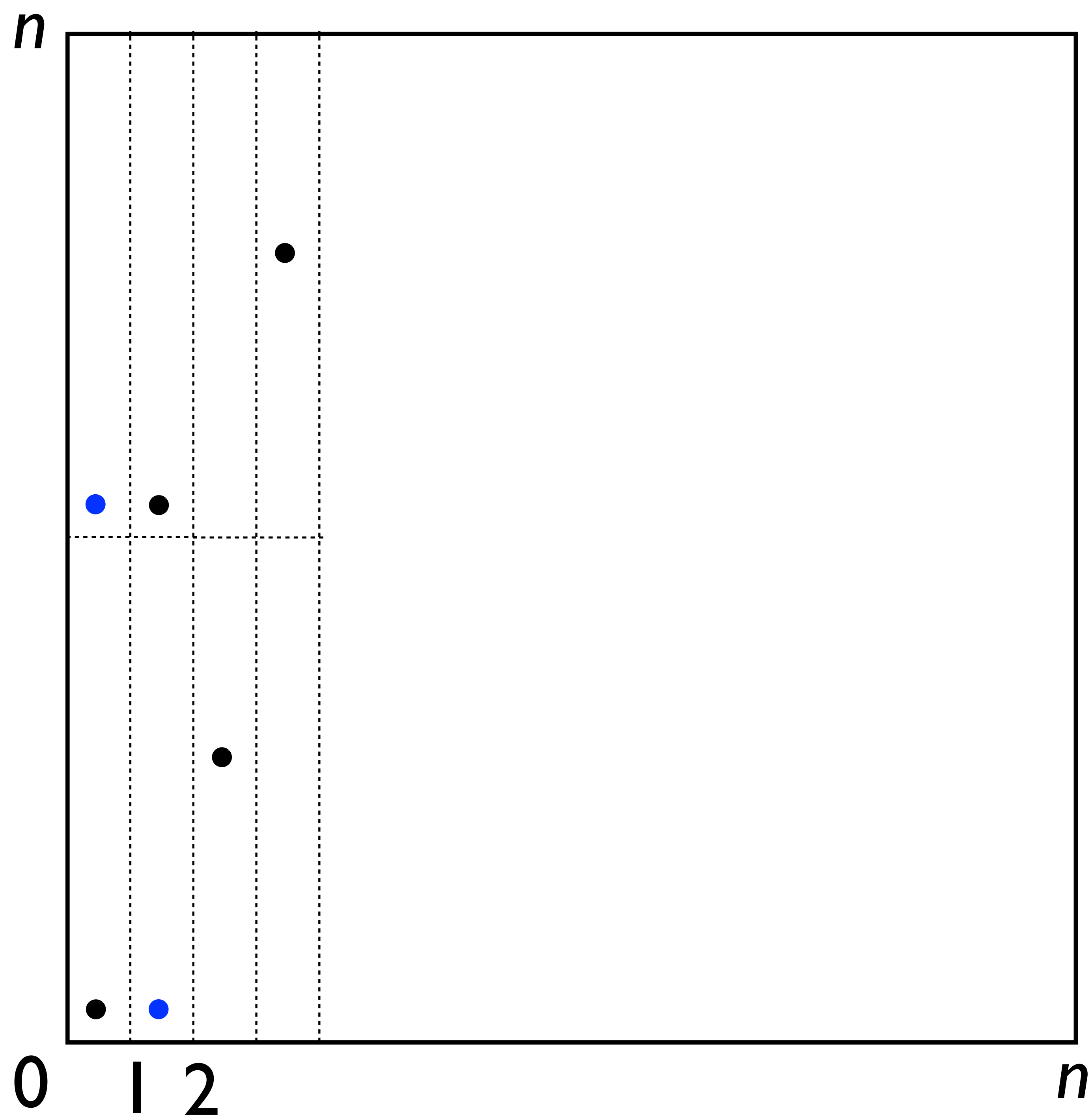

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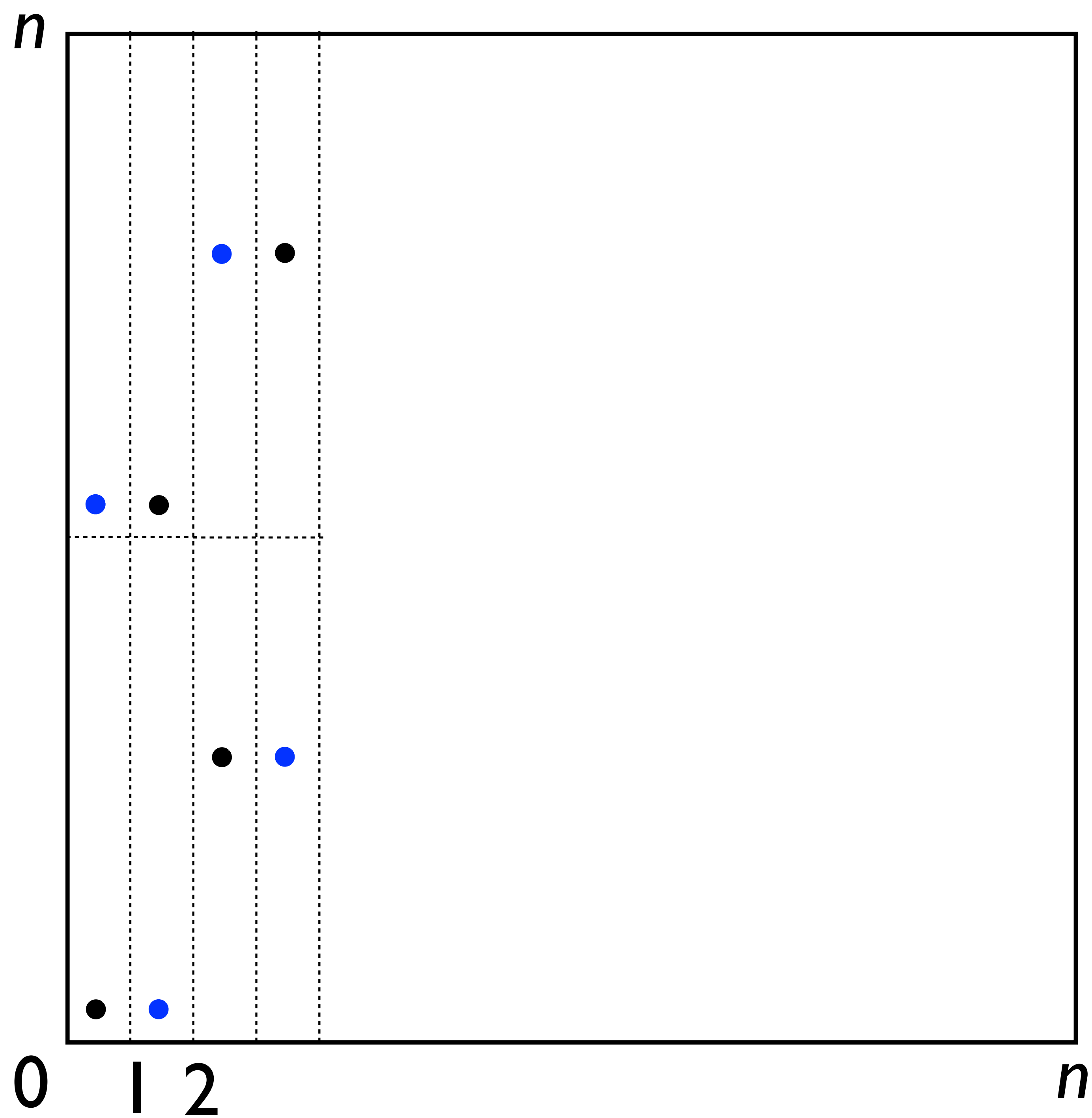

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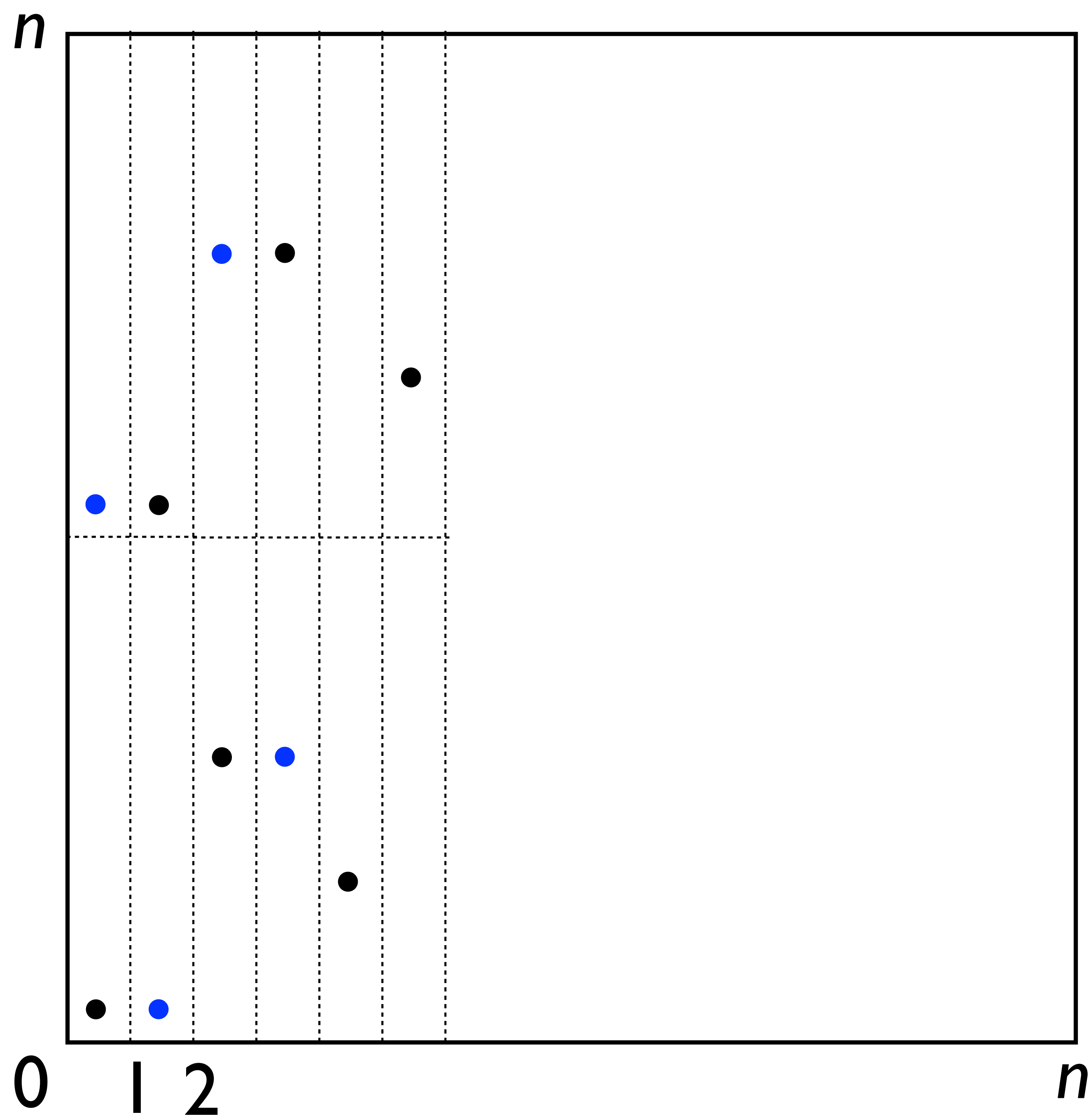

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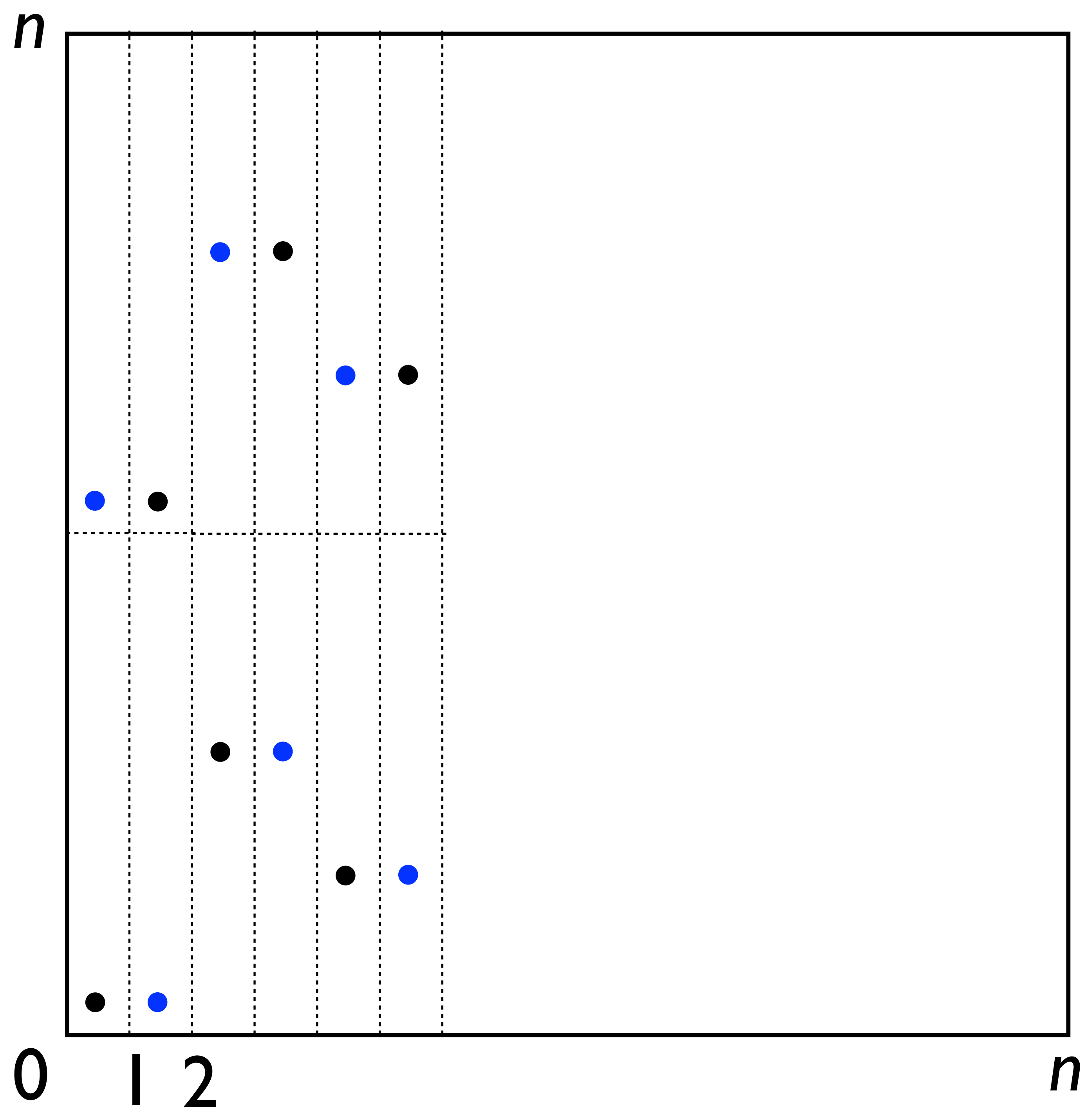

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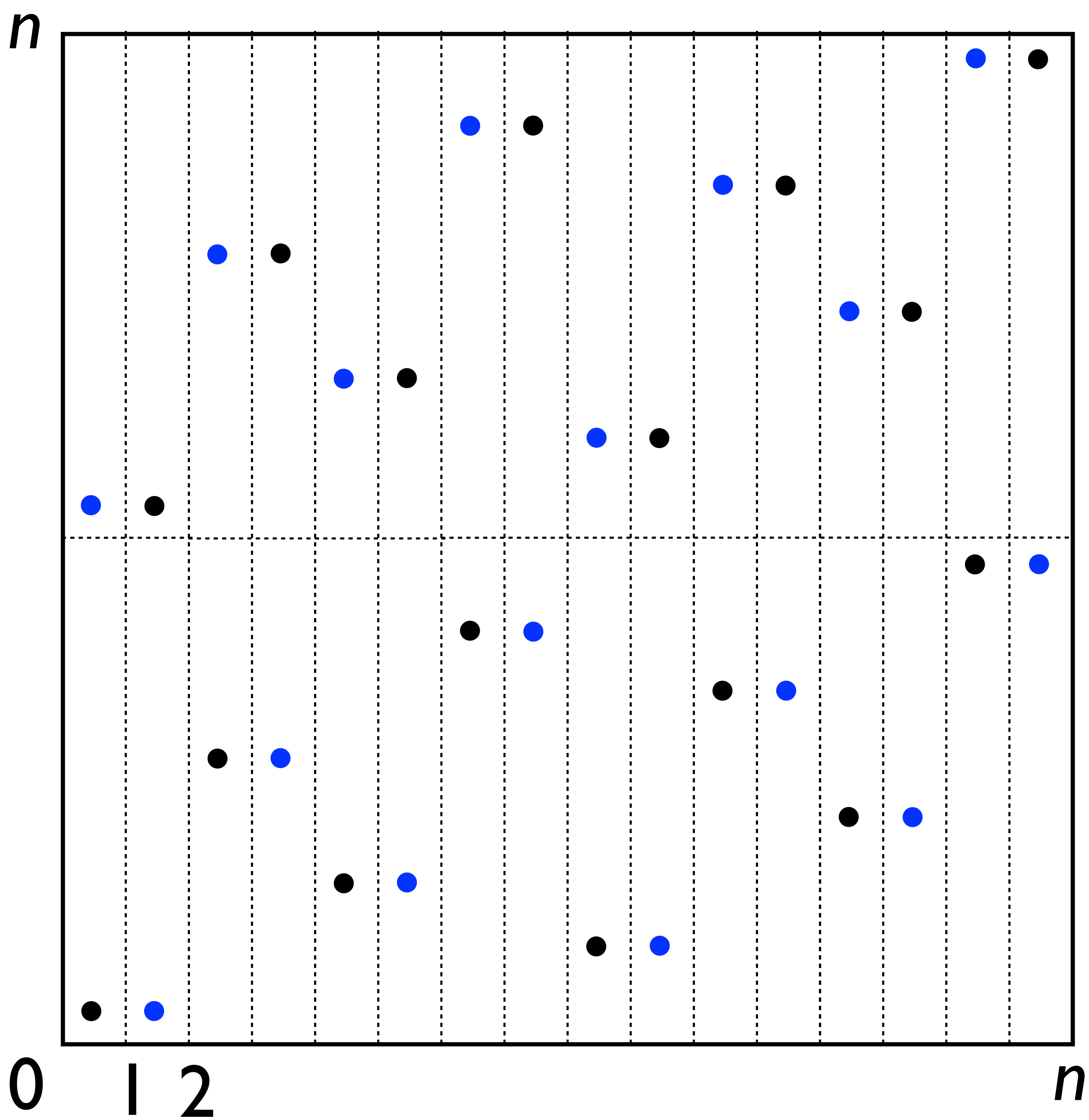

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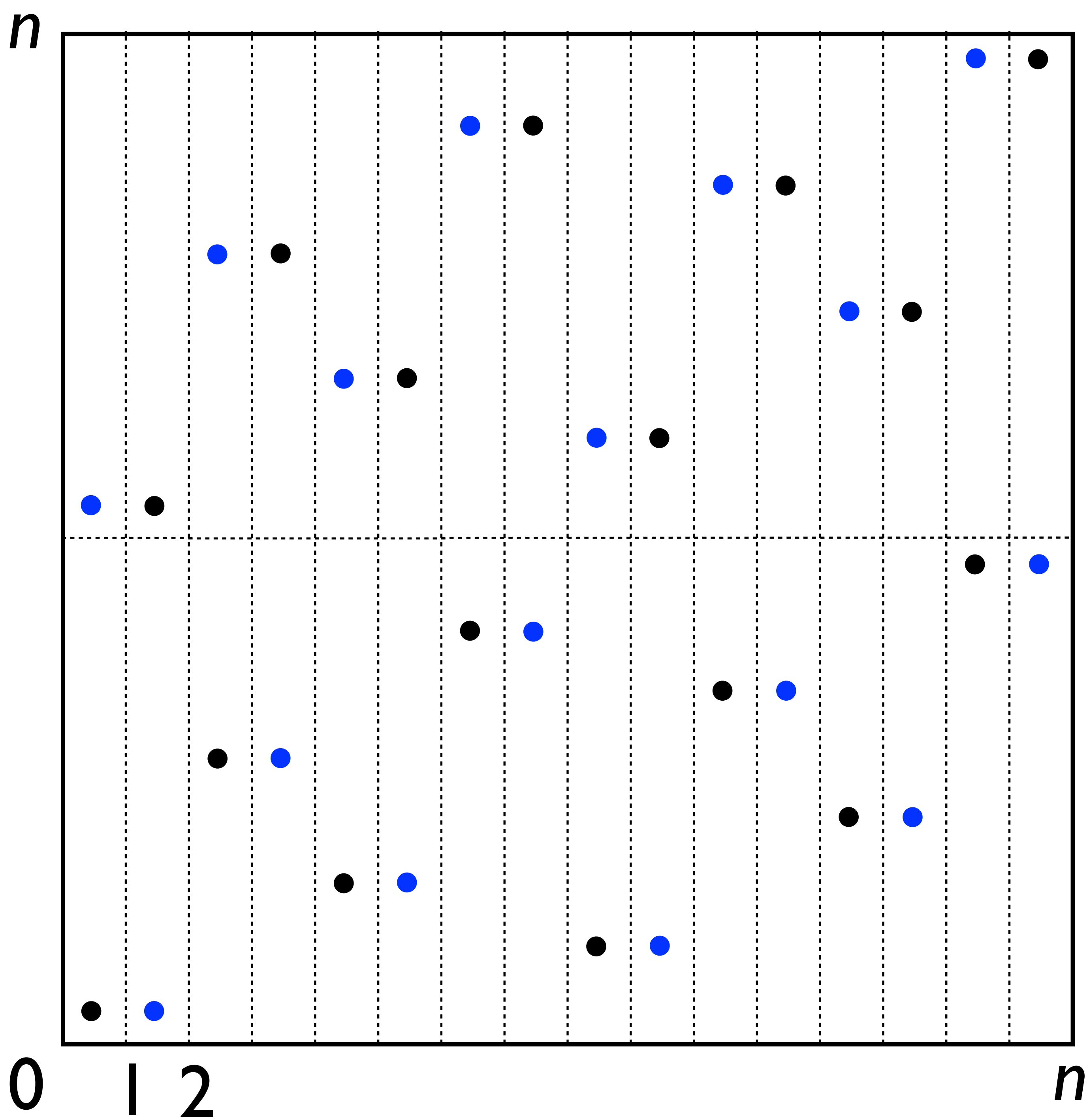

$2^{\frac{n}{2}}$ choices

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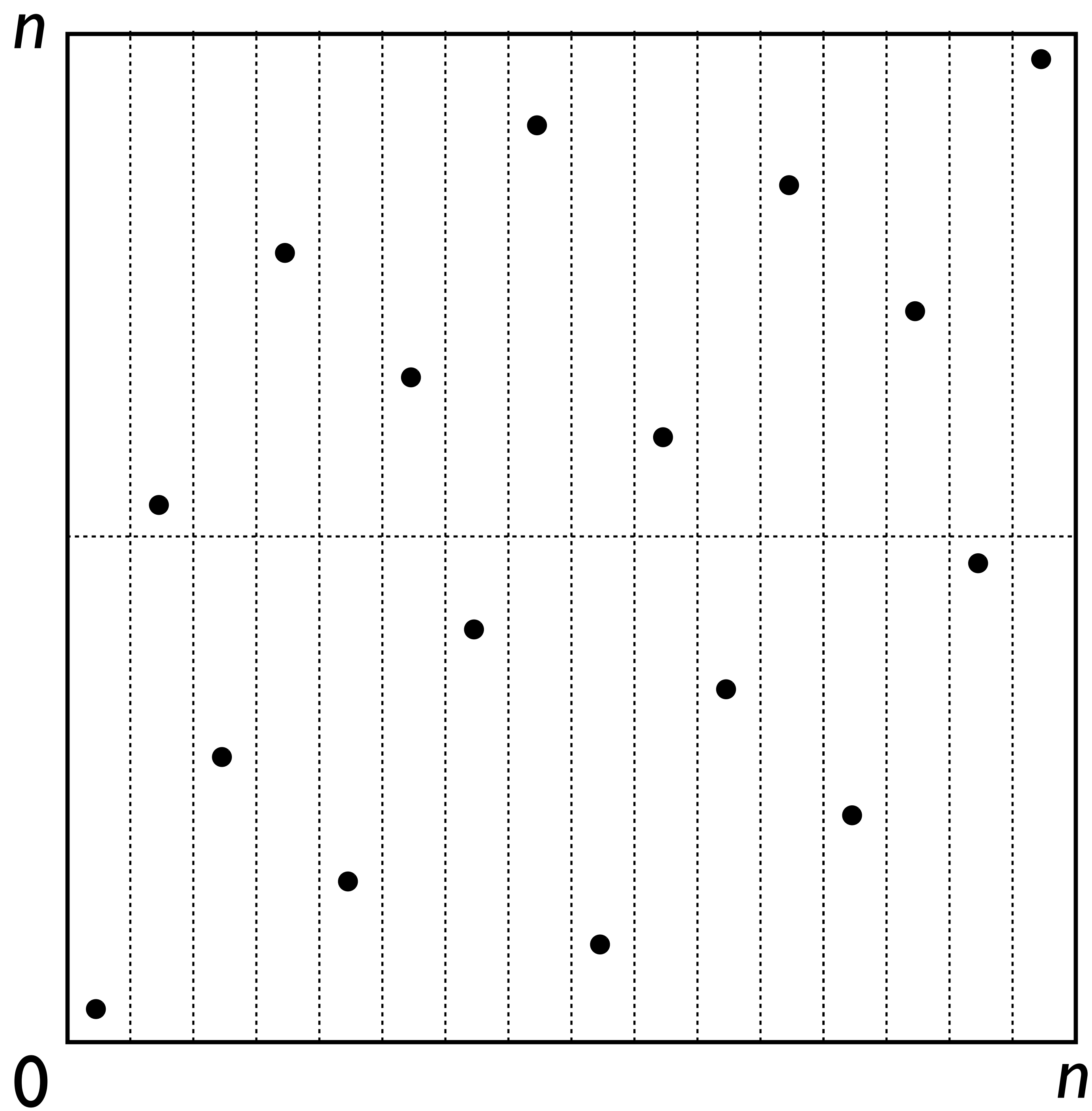

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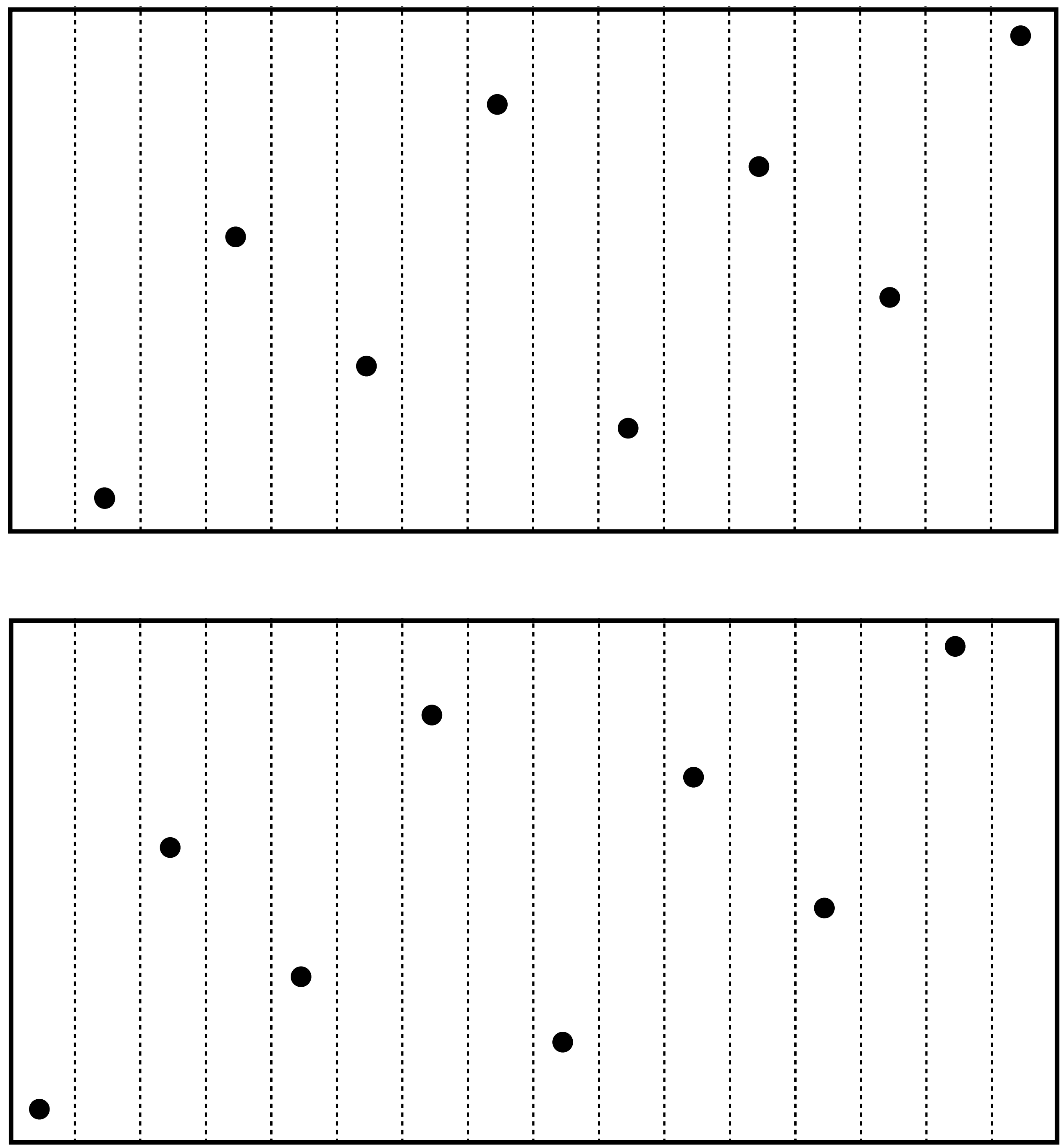

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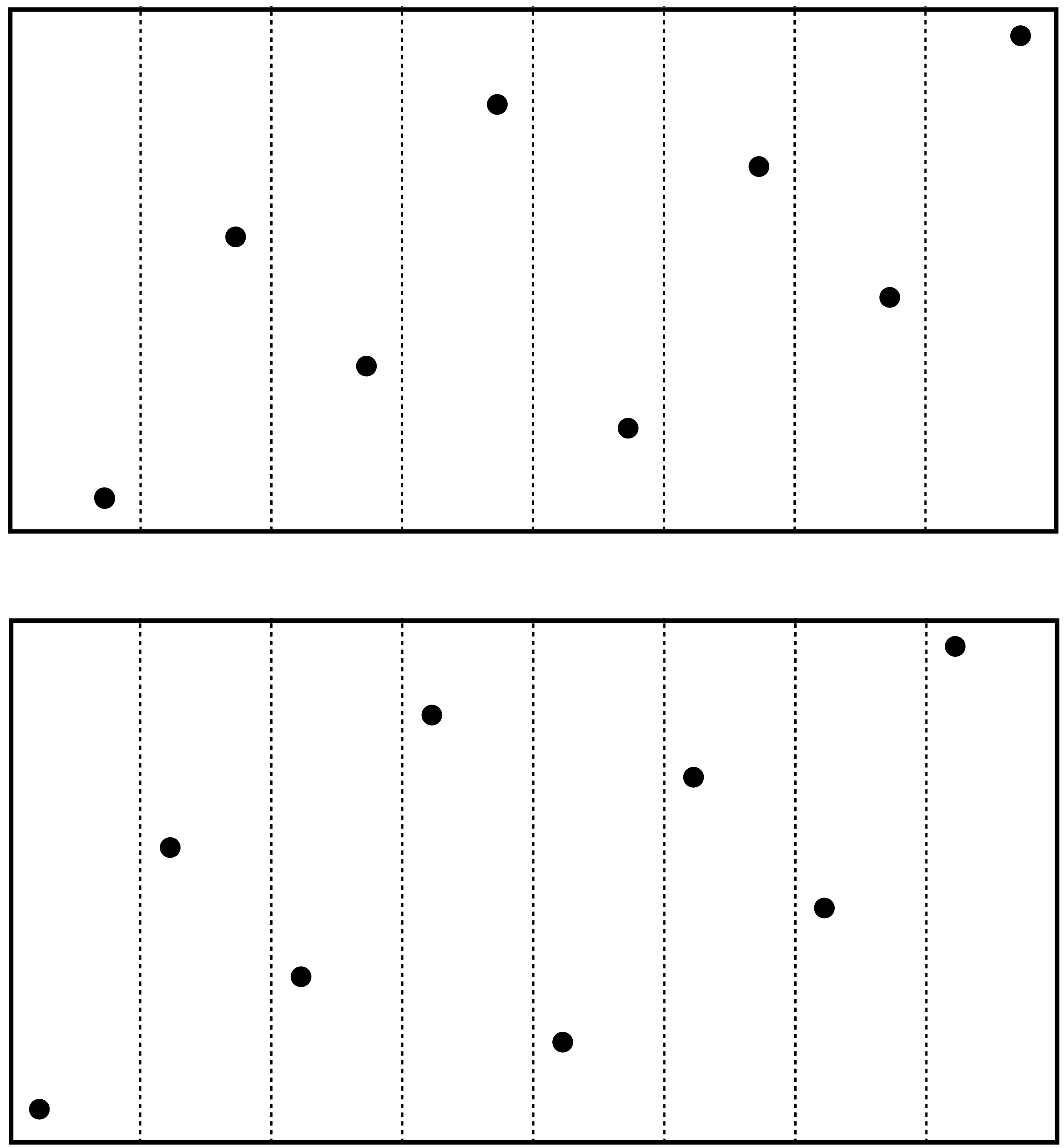

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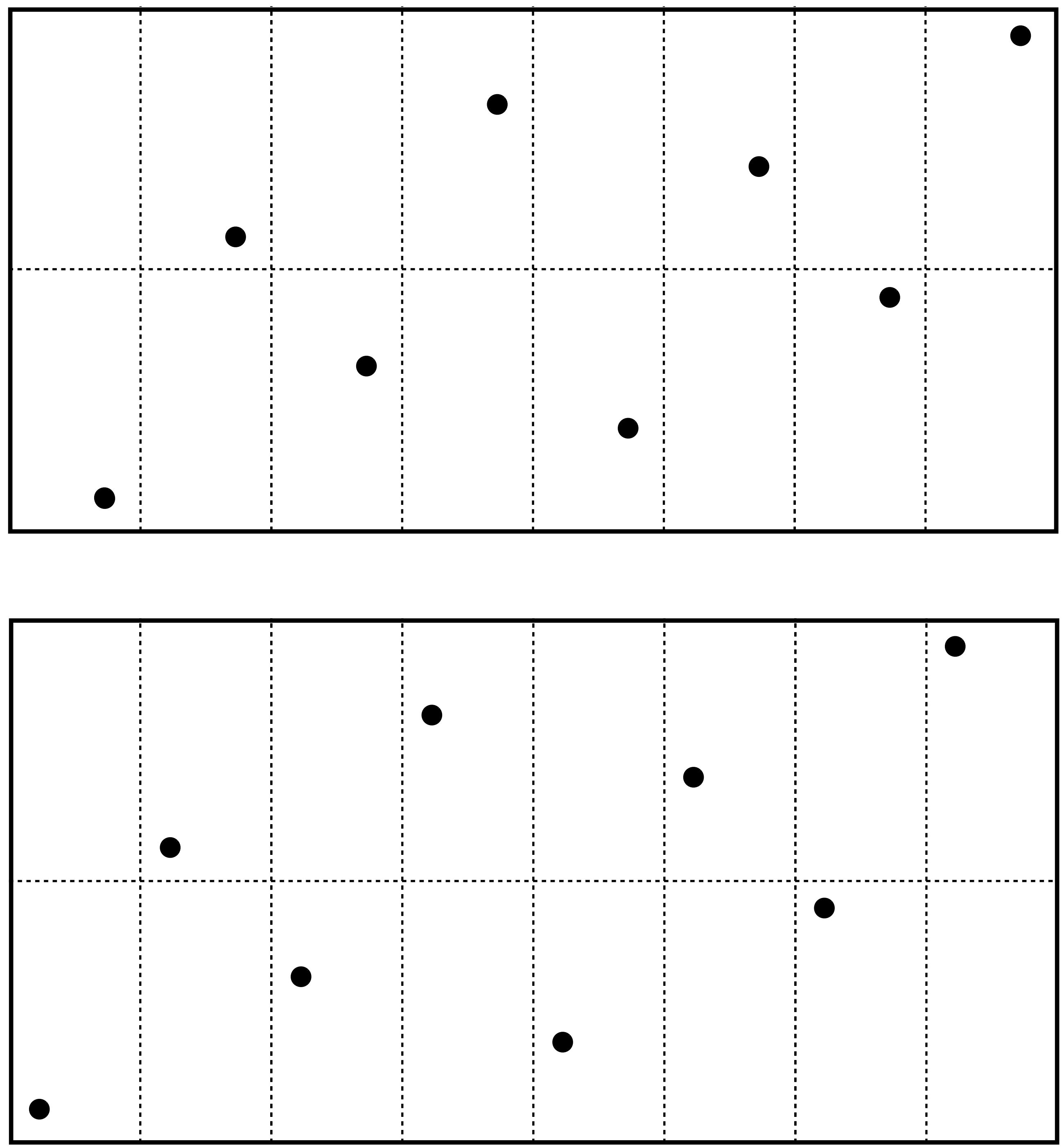

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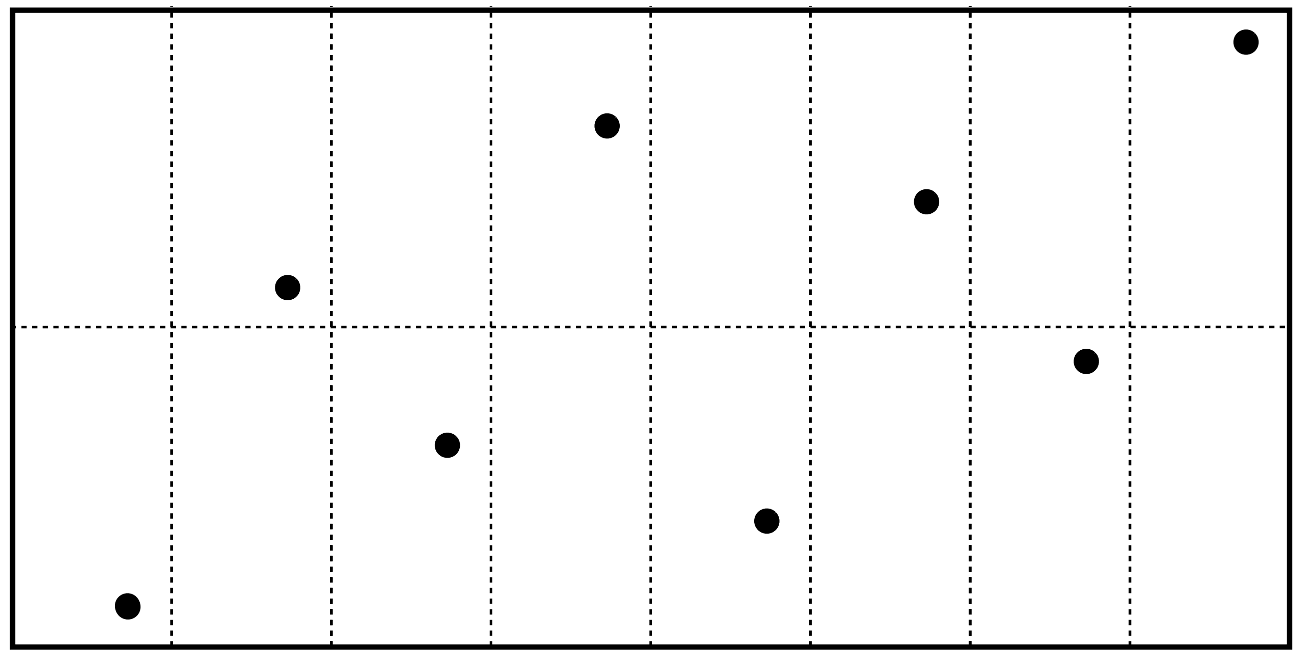
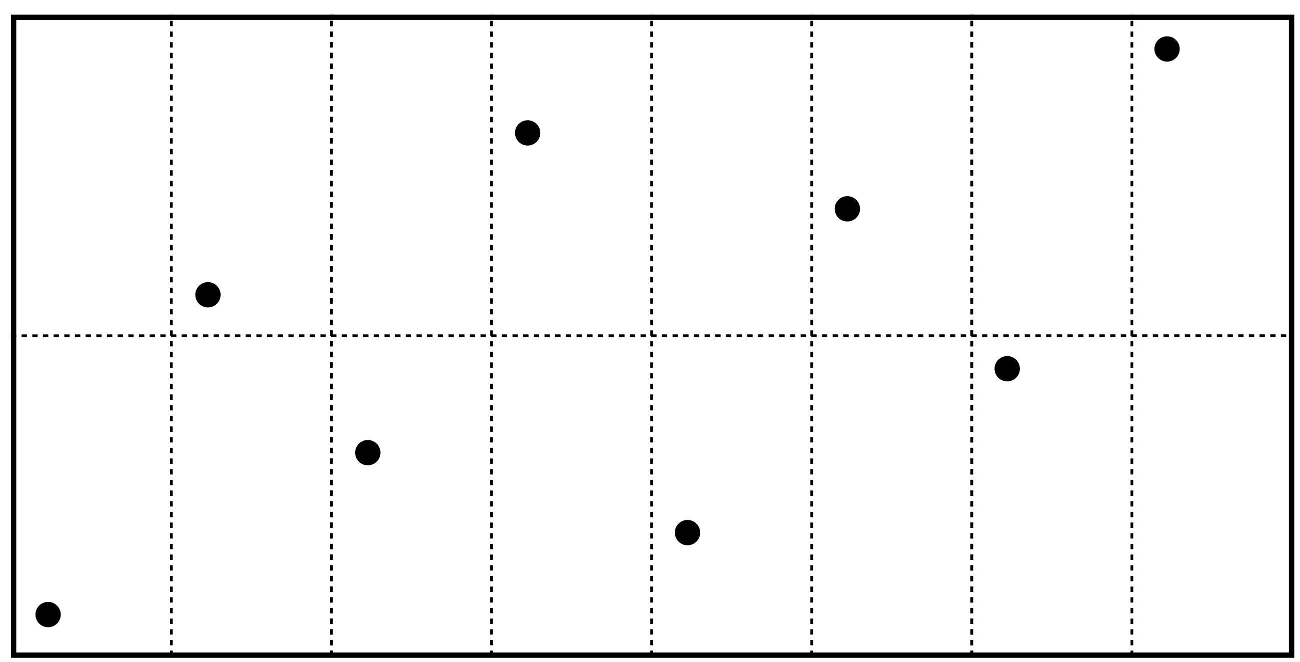


$2^{\frac{n}{4}}$ choices



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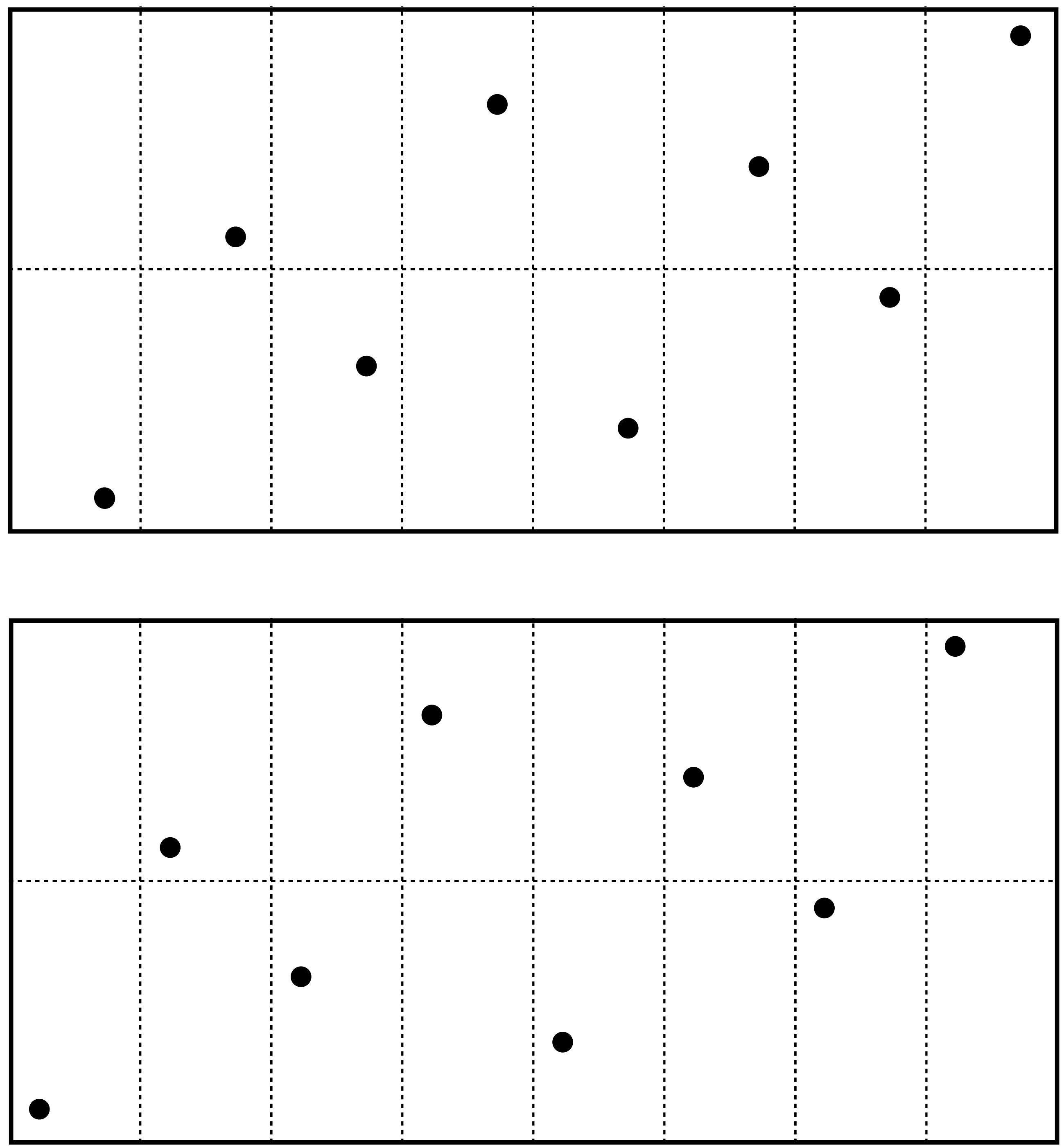

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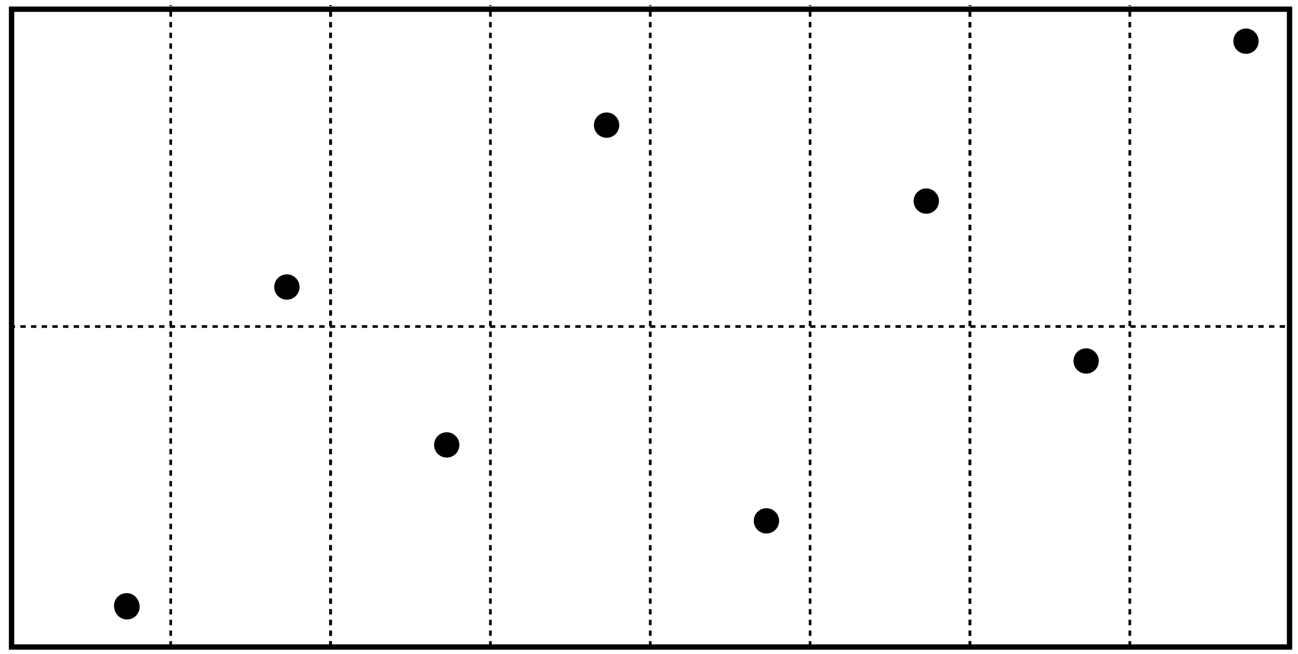

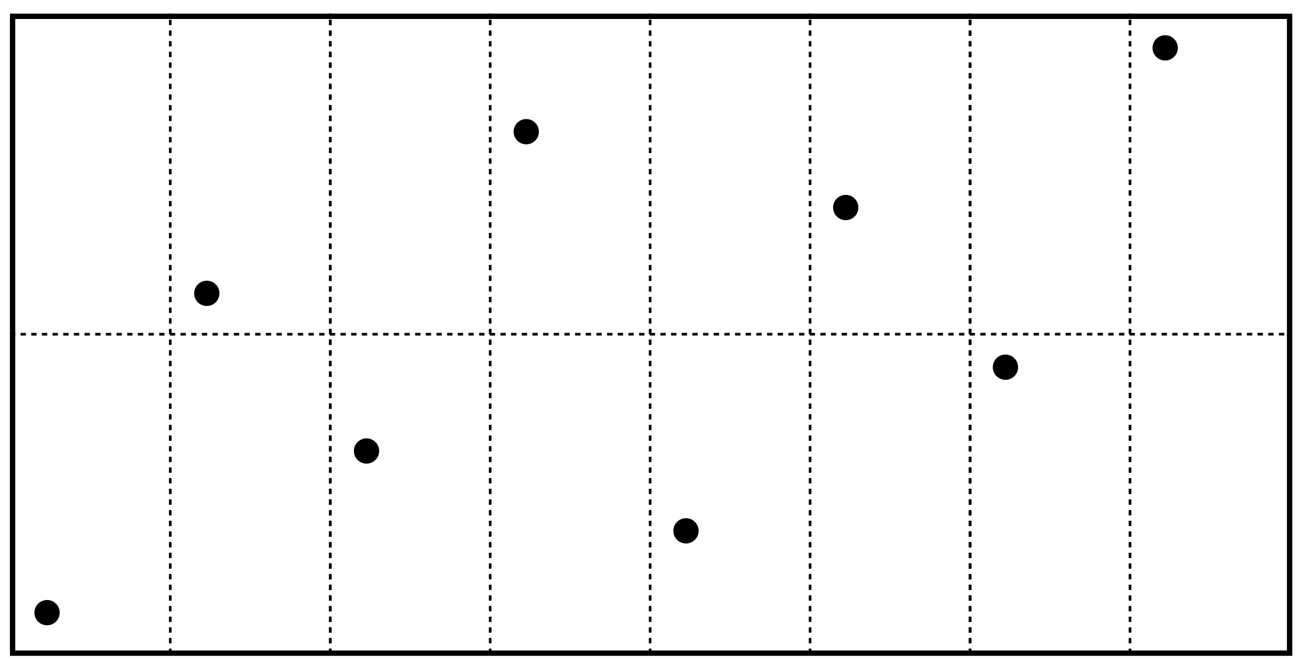

$2^{\frac{n}{2}}$ choices

$\log n$ layers $\Rightarrow 2^{\frac{n}{2}\log n}$ point sets.

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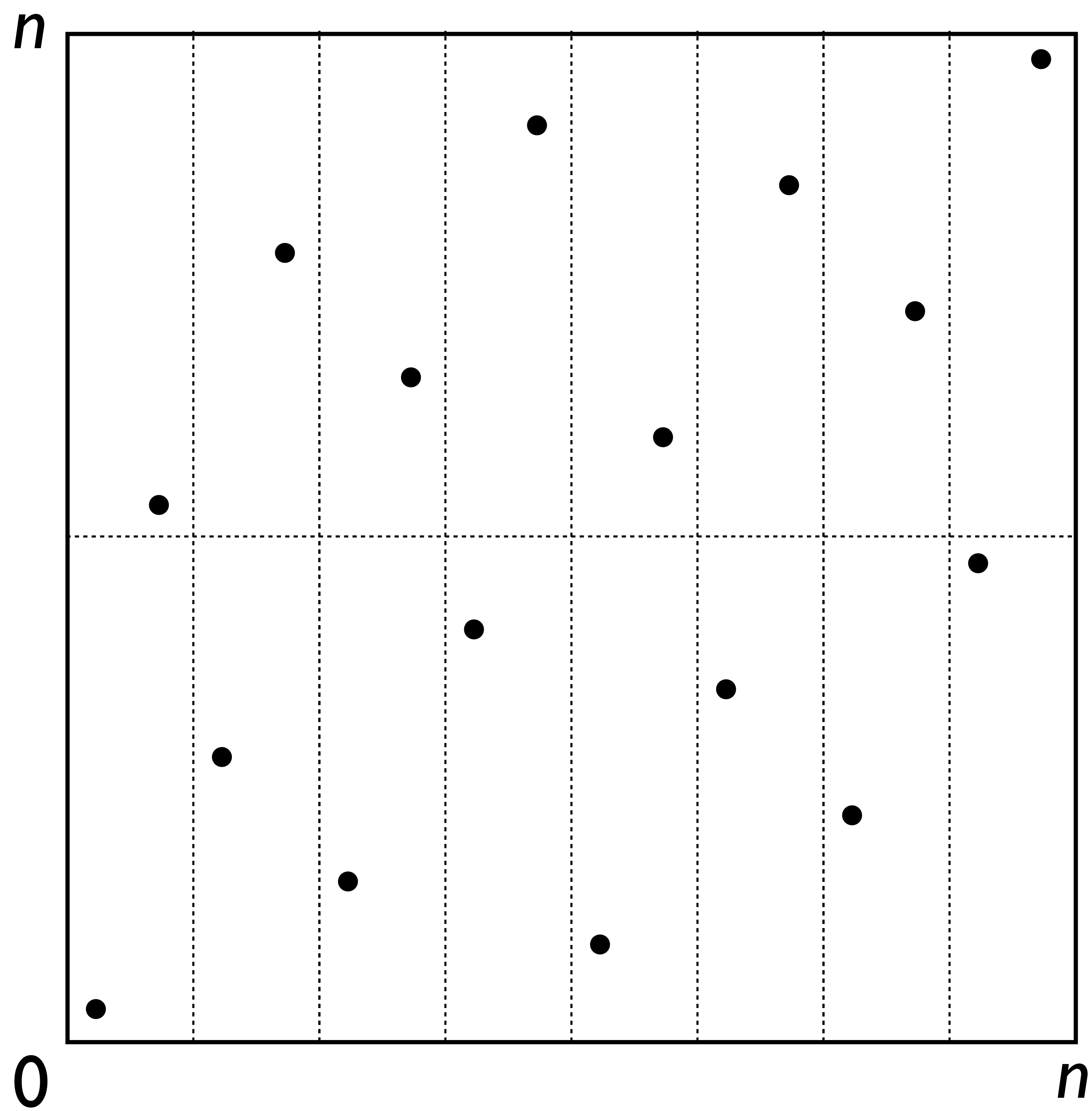
Corner Volume



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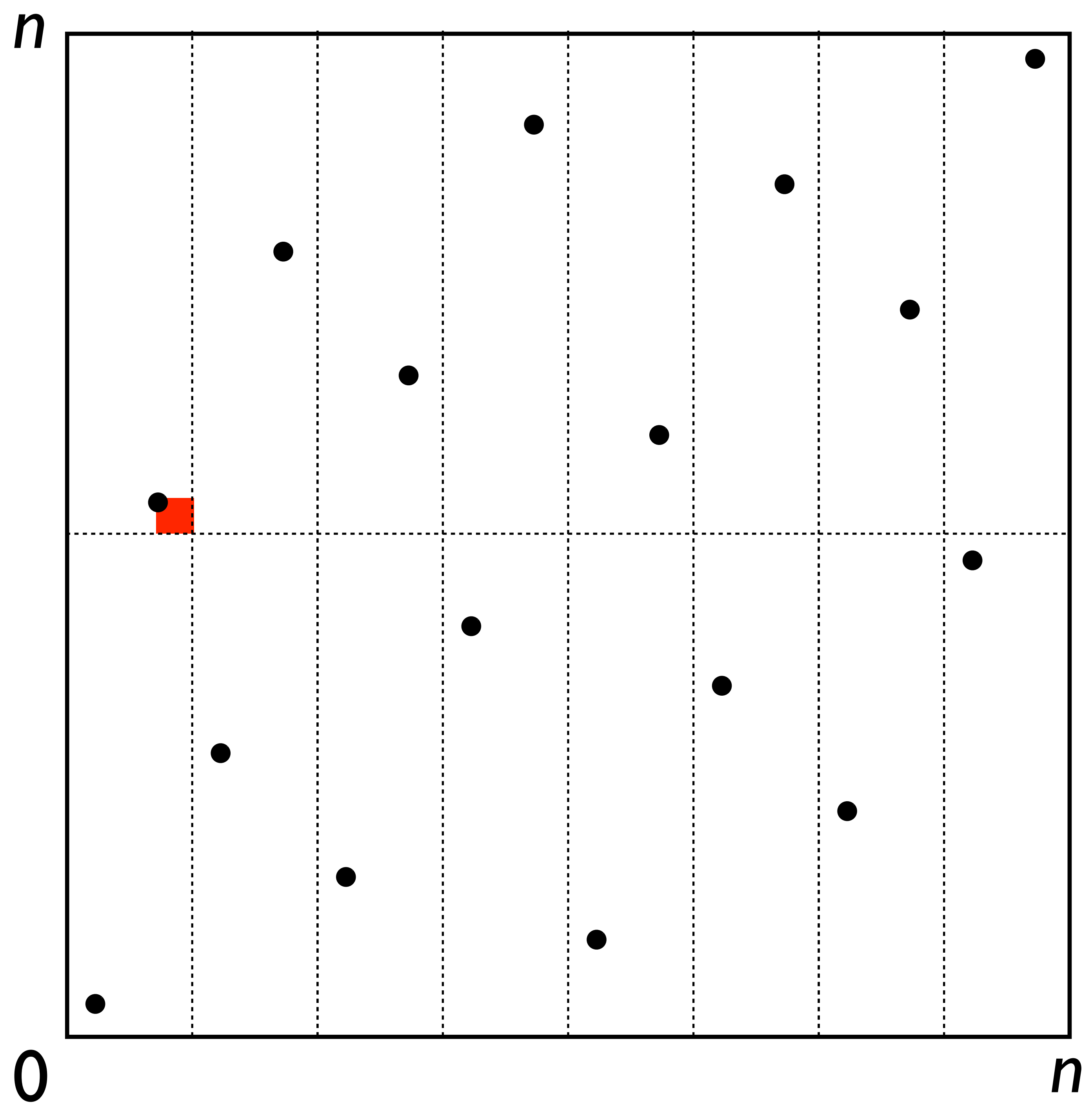

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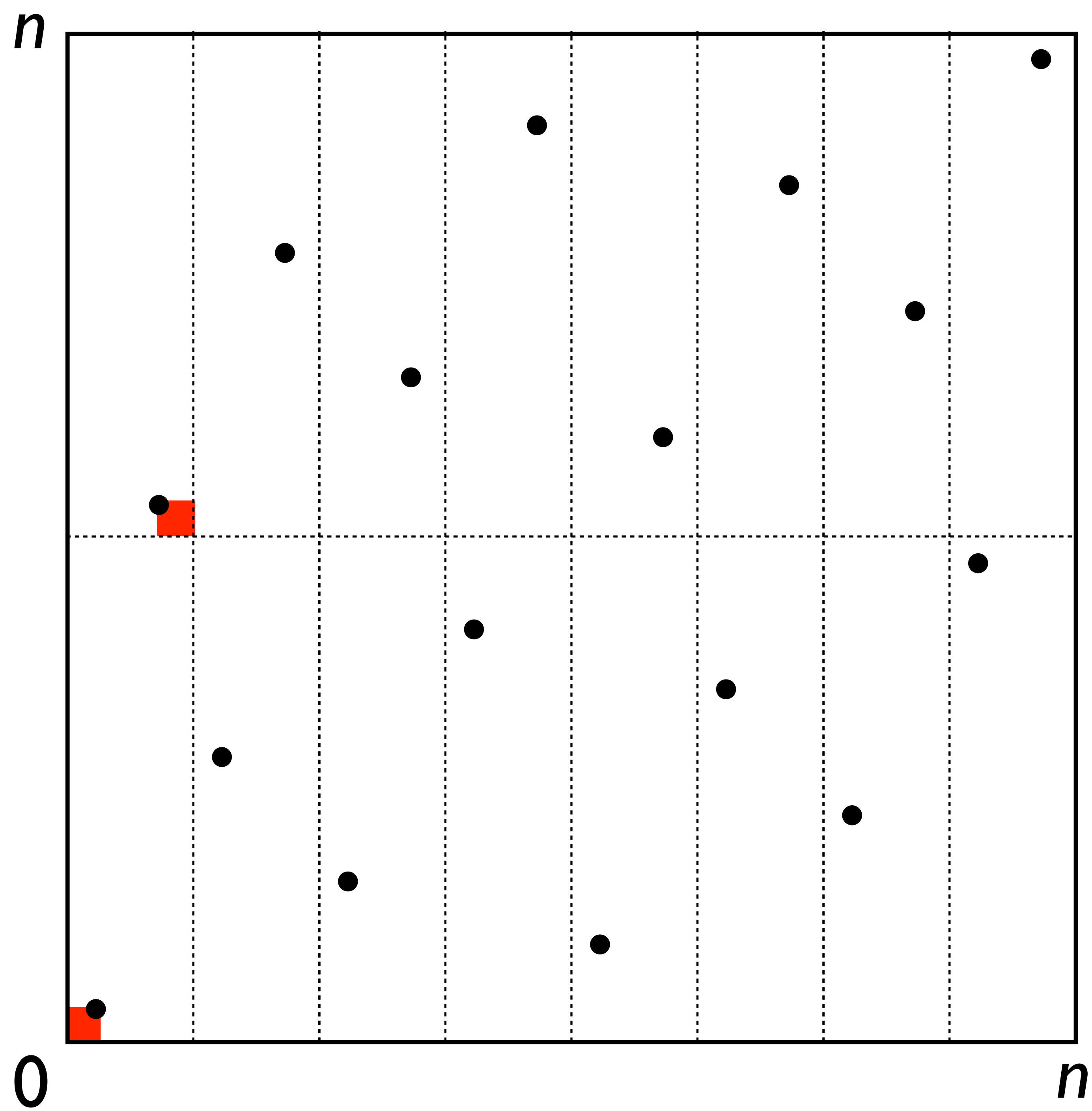
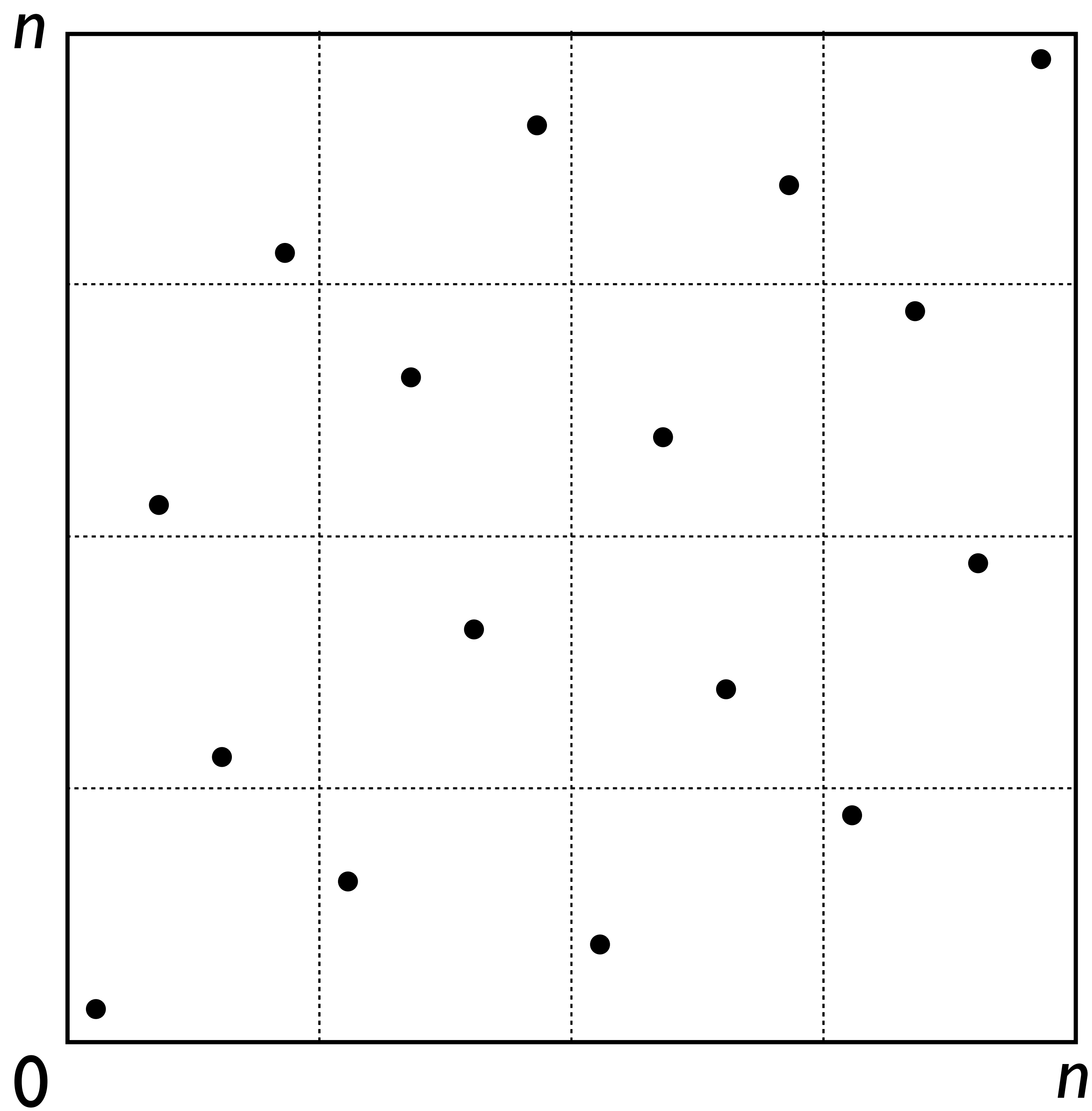


Corner Volume Distance



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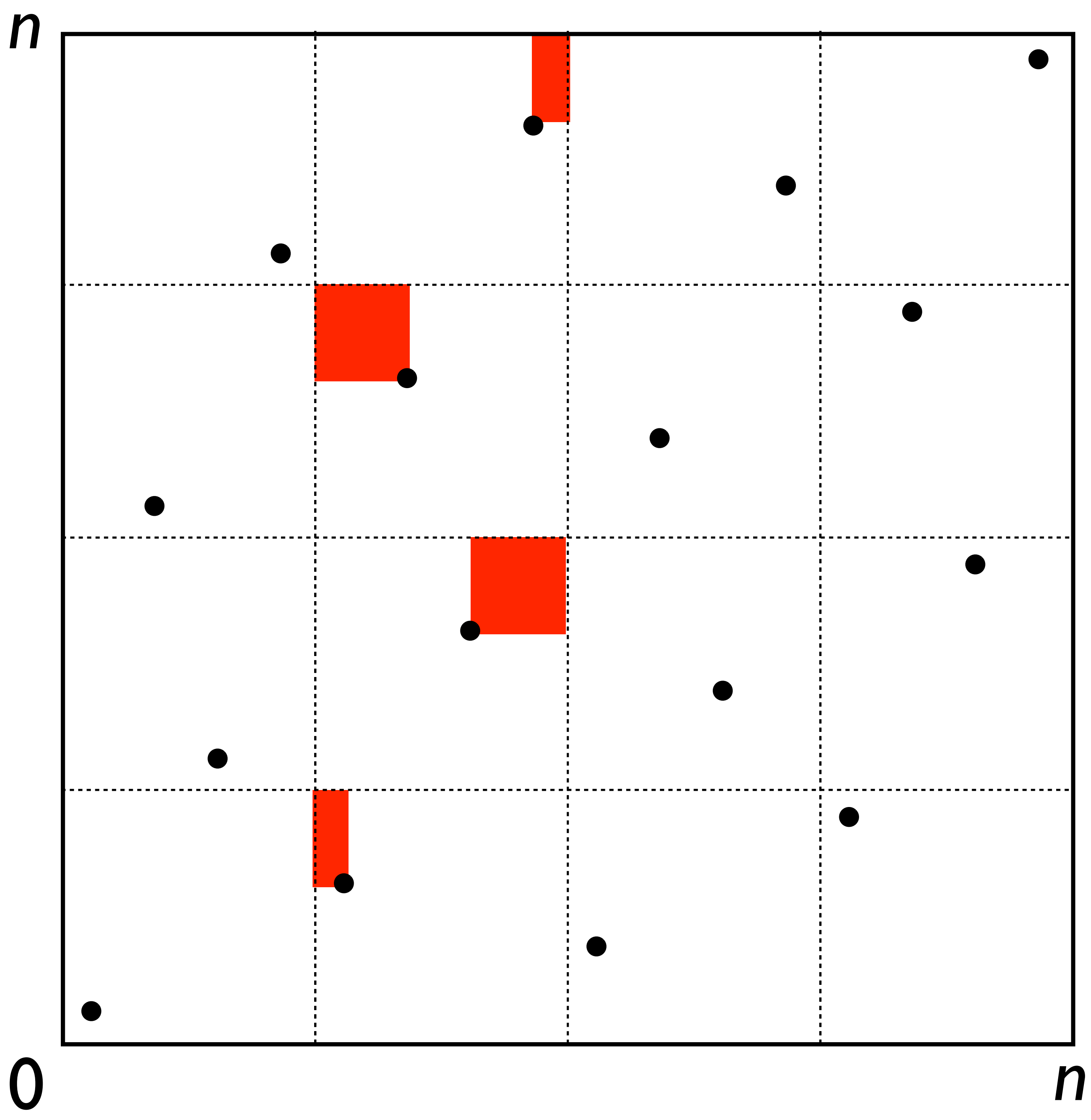

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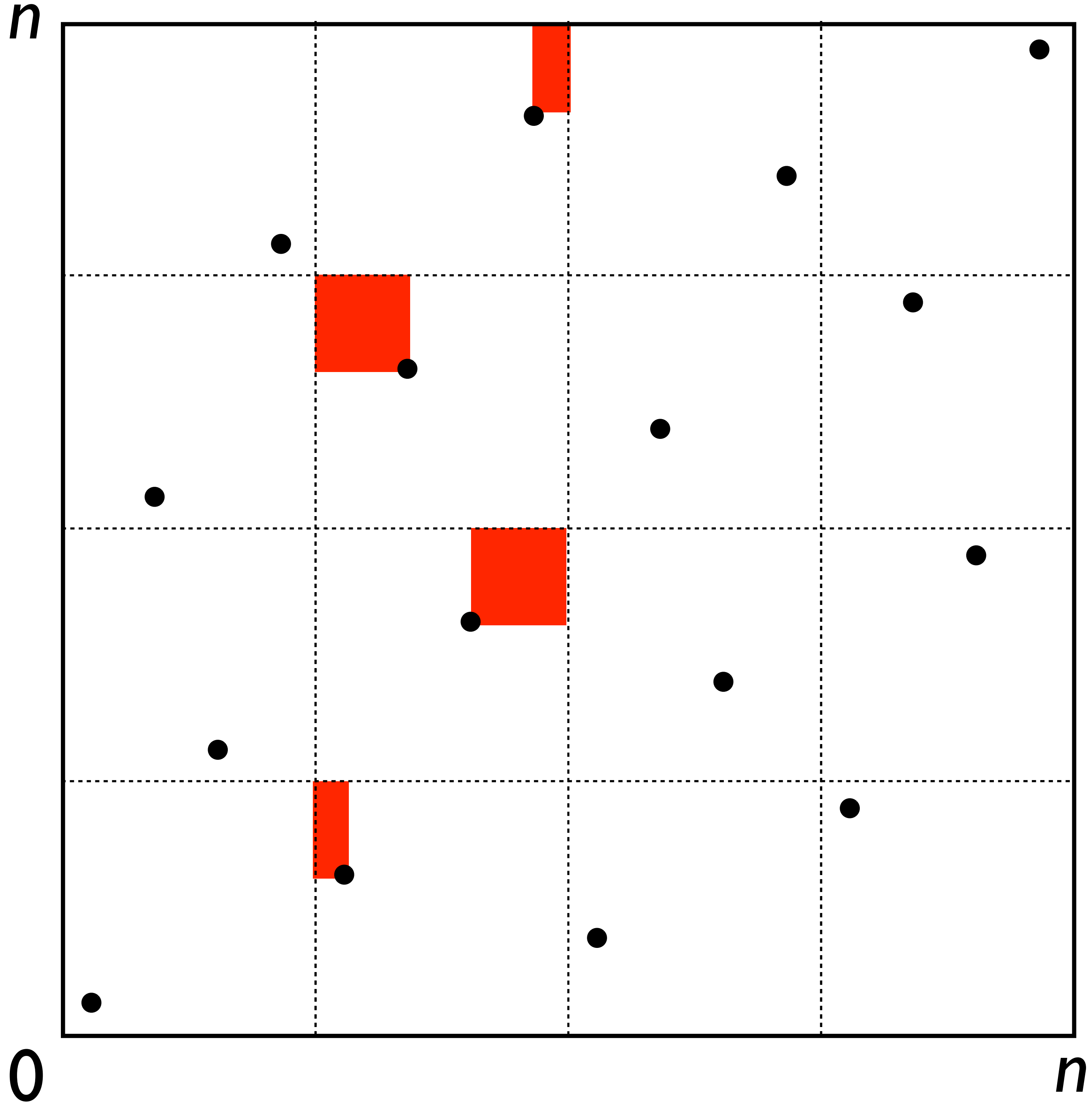

$n \log n$ corner volumes

Corner Volume

Lemma: If corner volume sum $\geq cn^2 \log n$, then $\mathrm{disc}(P) = \Omega(\log n)$.

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- Widely used for proving lower bound for Lebesgue discrepancy.

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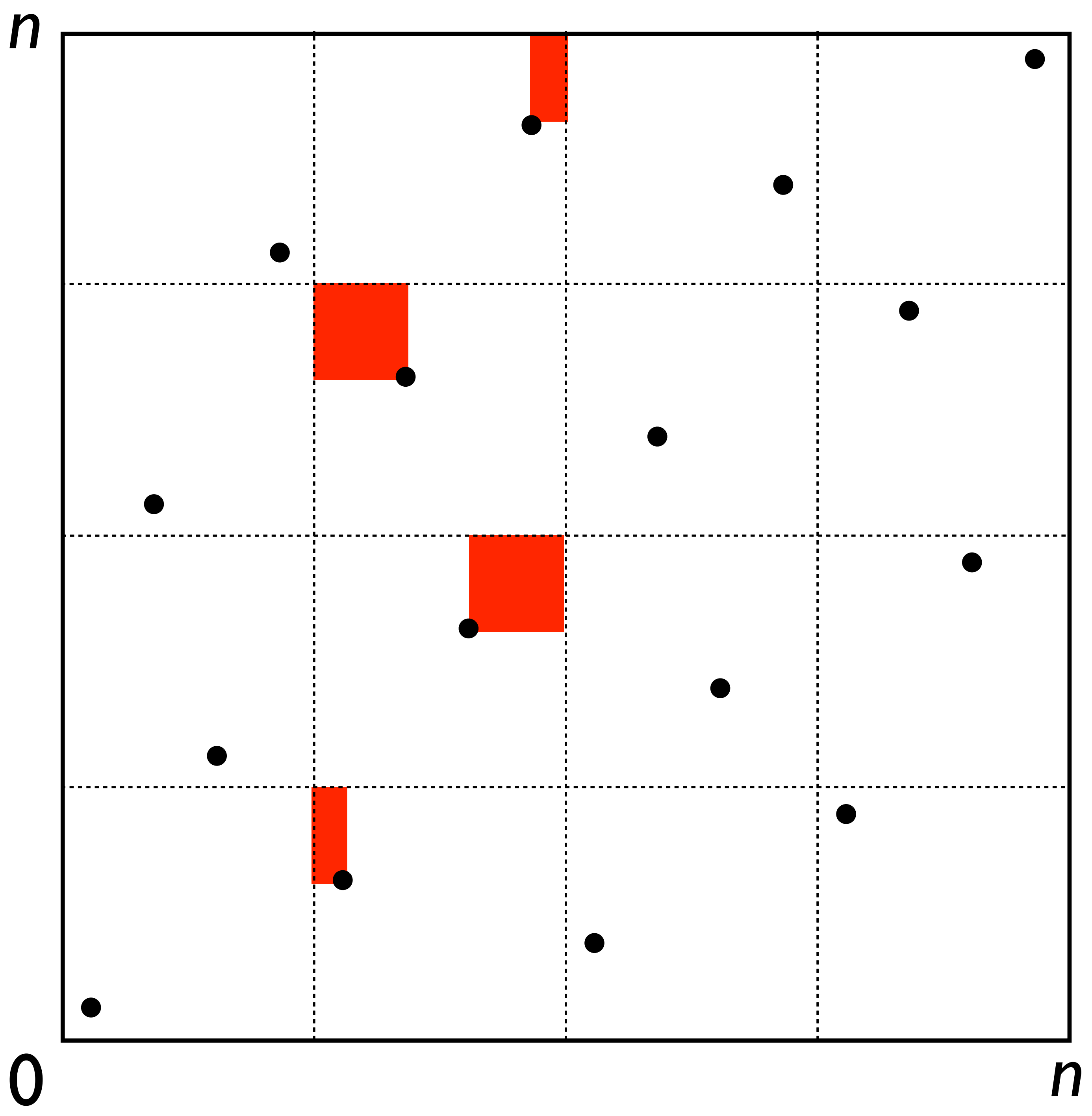
- Constant fraction of cells in each layer have corner volume $\Omega(n)$.

Proof: Roth's orthogonal function method (Roth 1954).

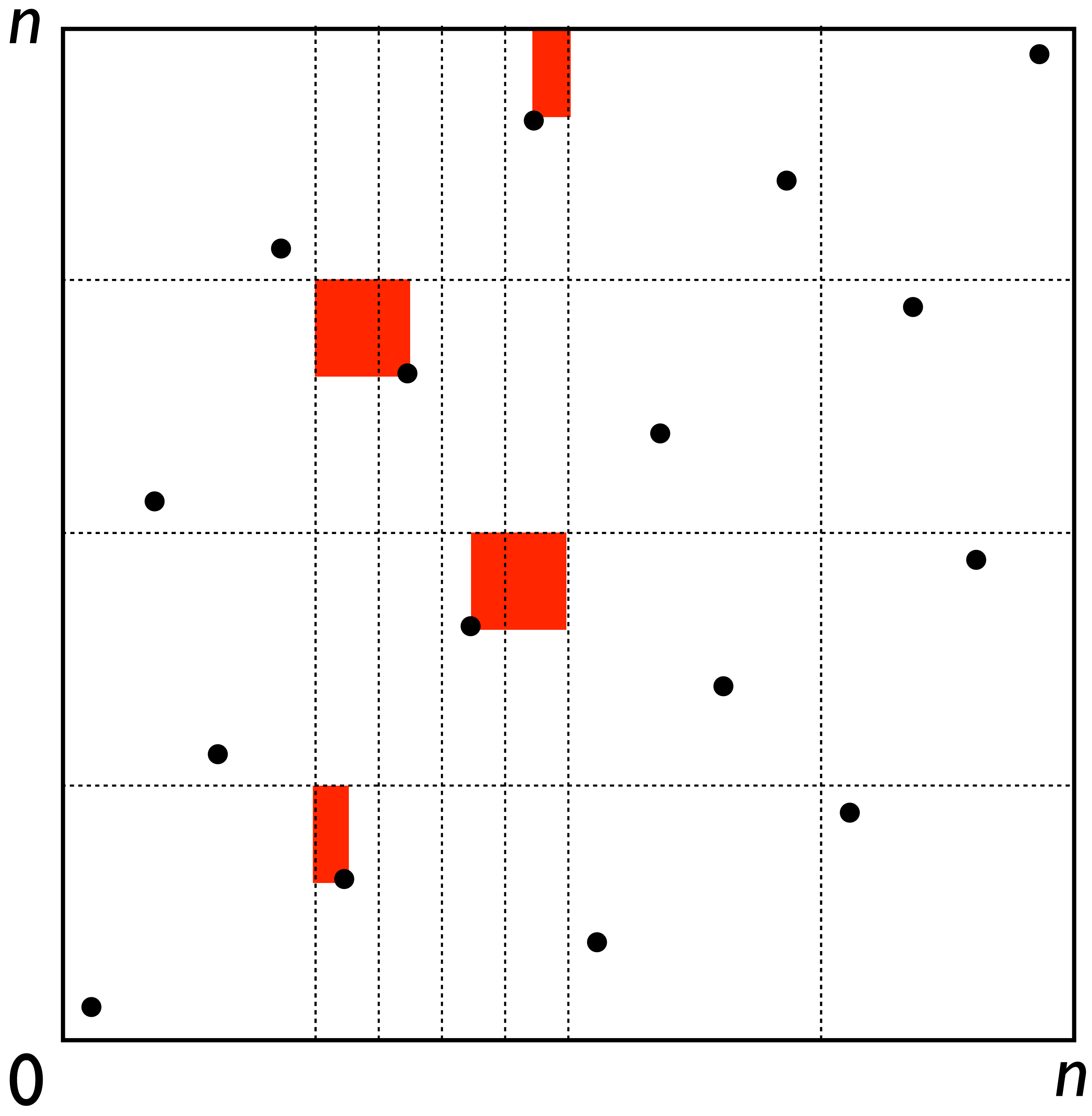
- Widely used for proving lower bound for Lebesgue discrepancy.

- Chazelle: "Most beautiful proof in discrepancy theory".

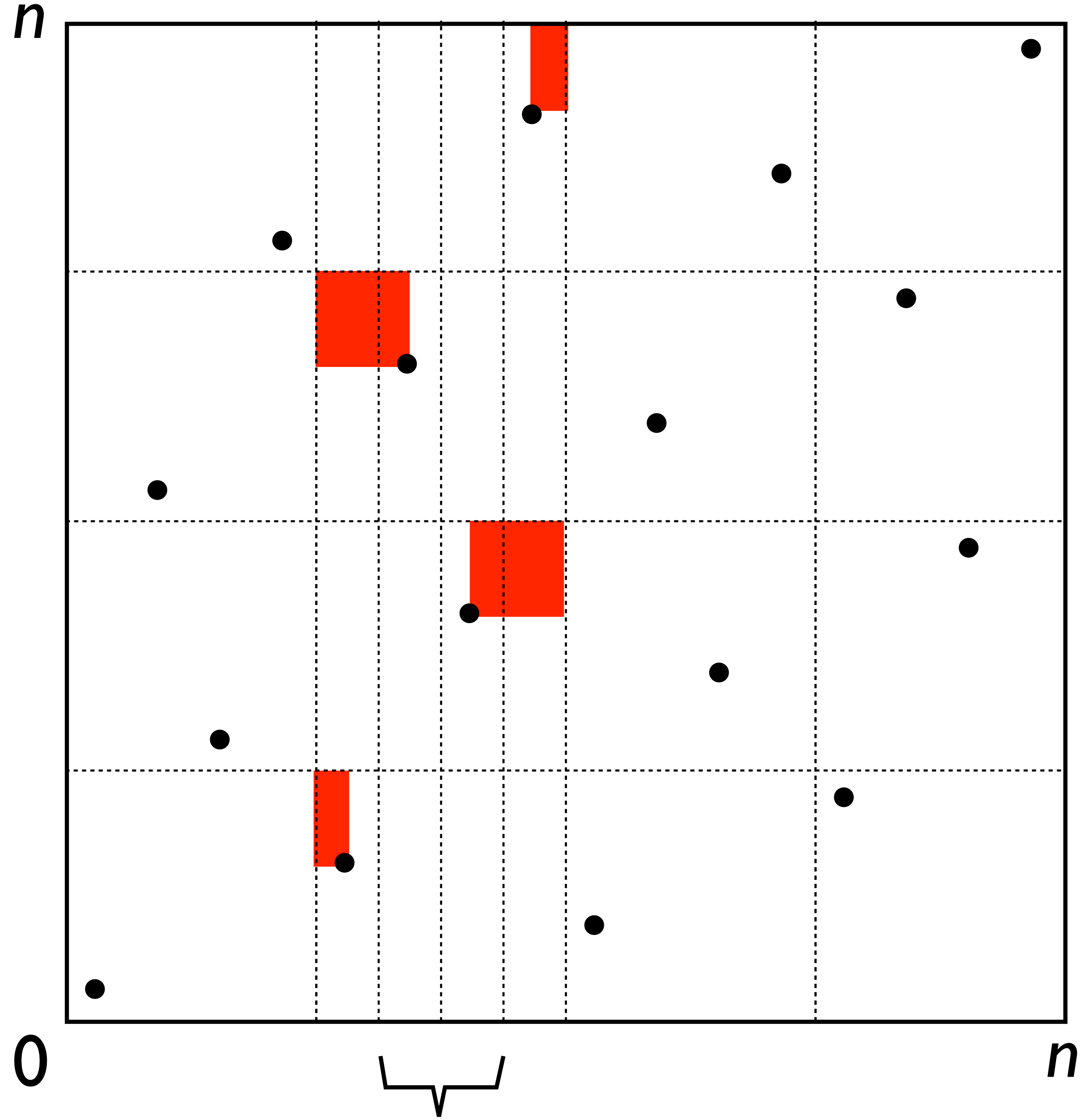
Large Corner Volume



Large Corner Volume

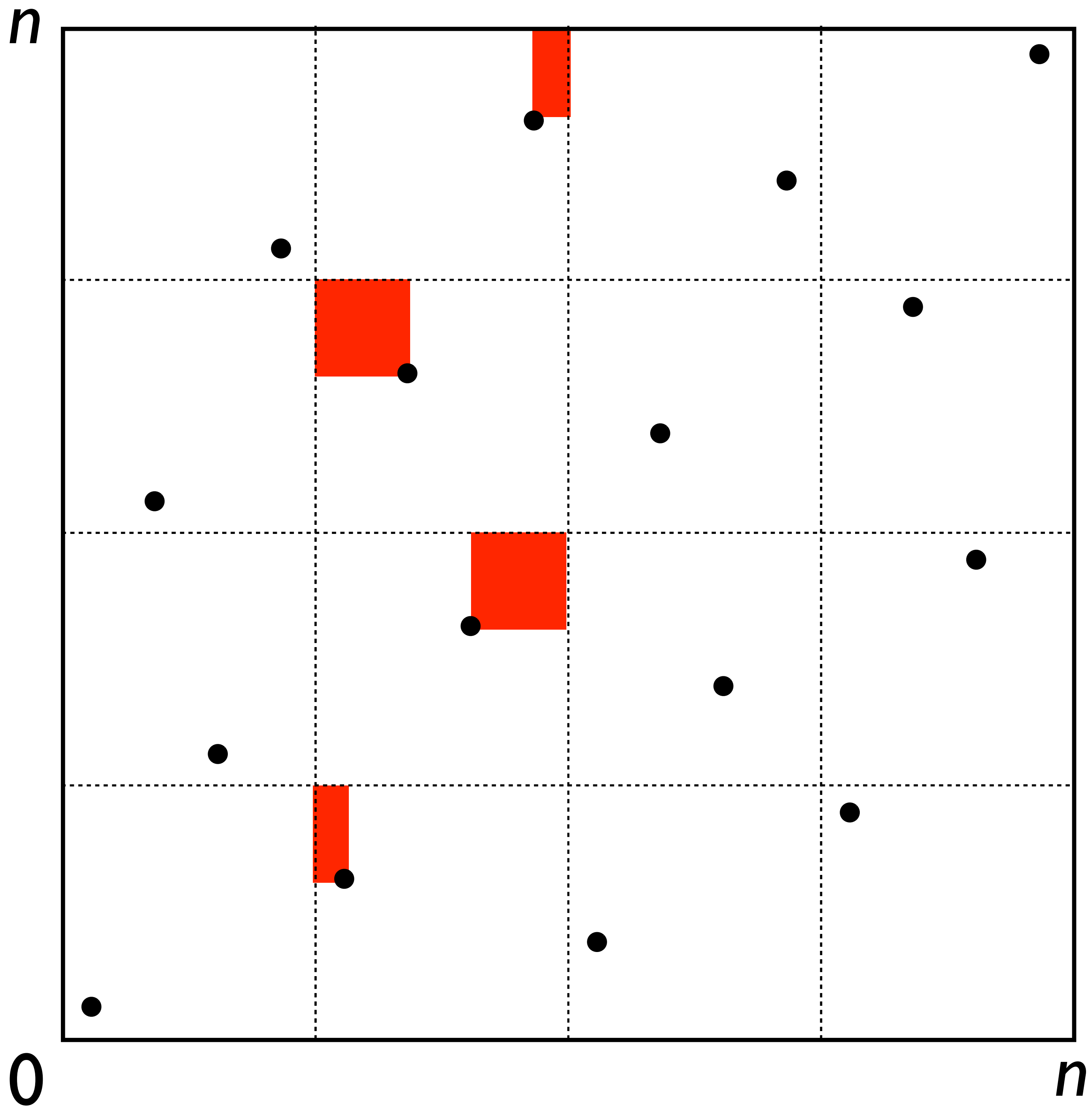


Large Corner Volume

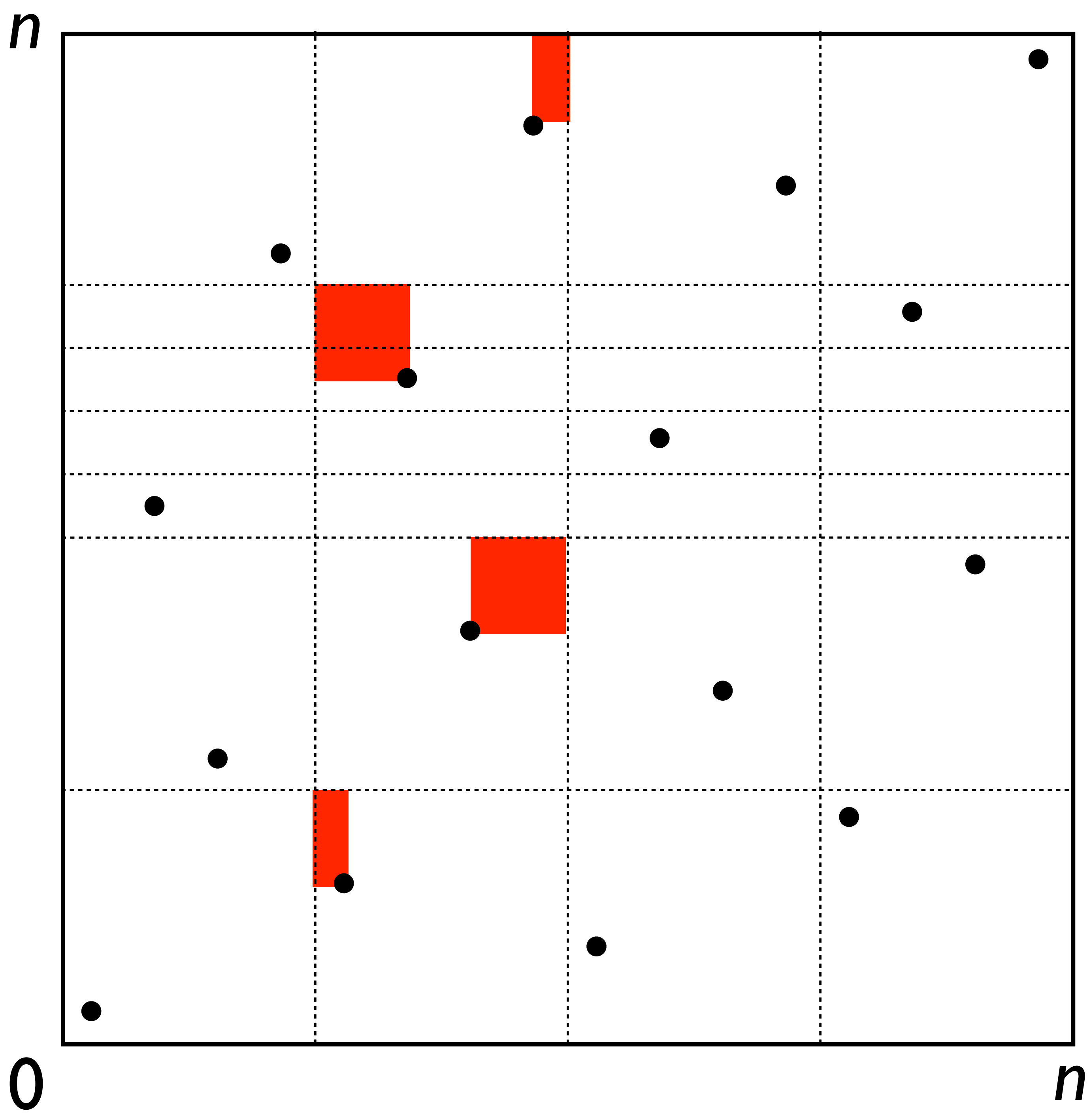


$\frac{3}{4}n$ points with $\geq \frac{1}{8} * x$ length.

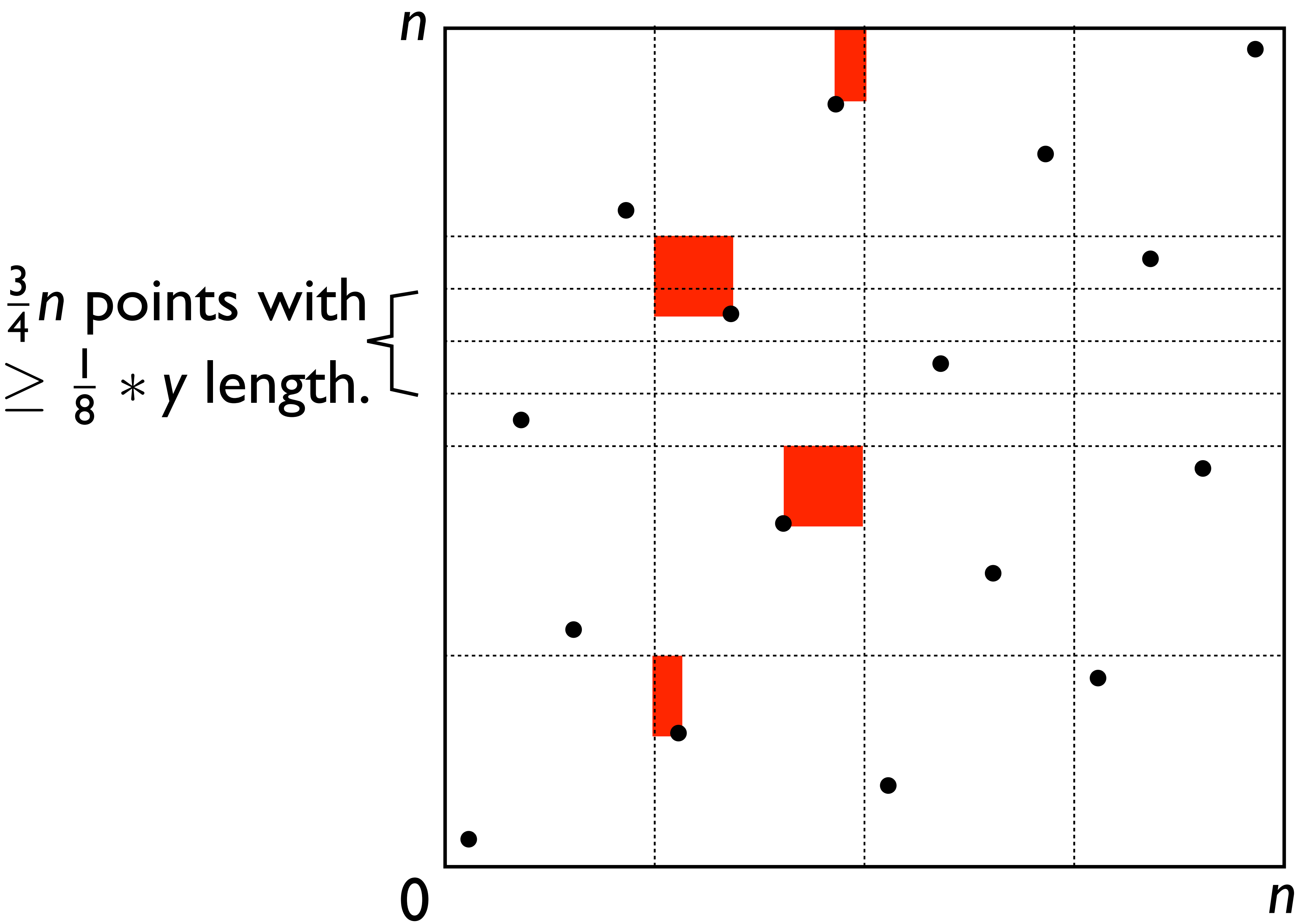
Large Corner Volume



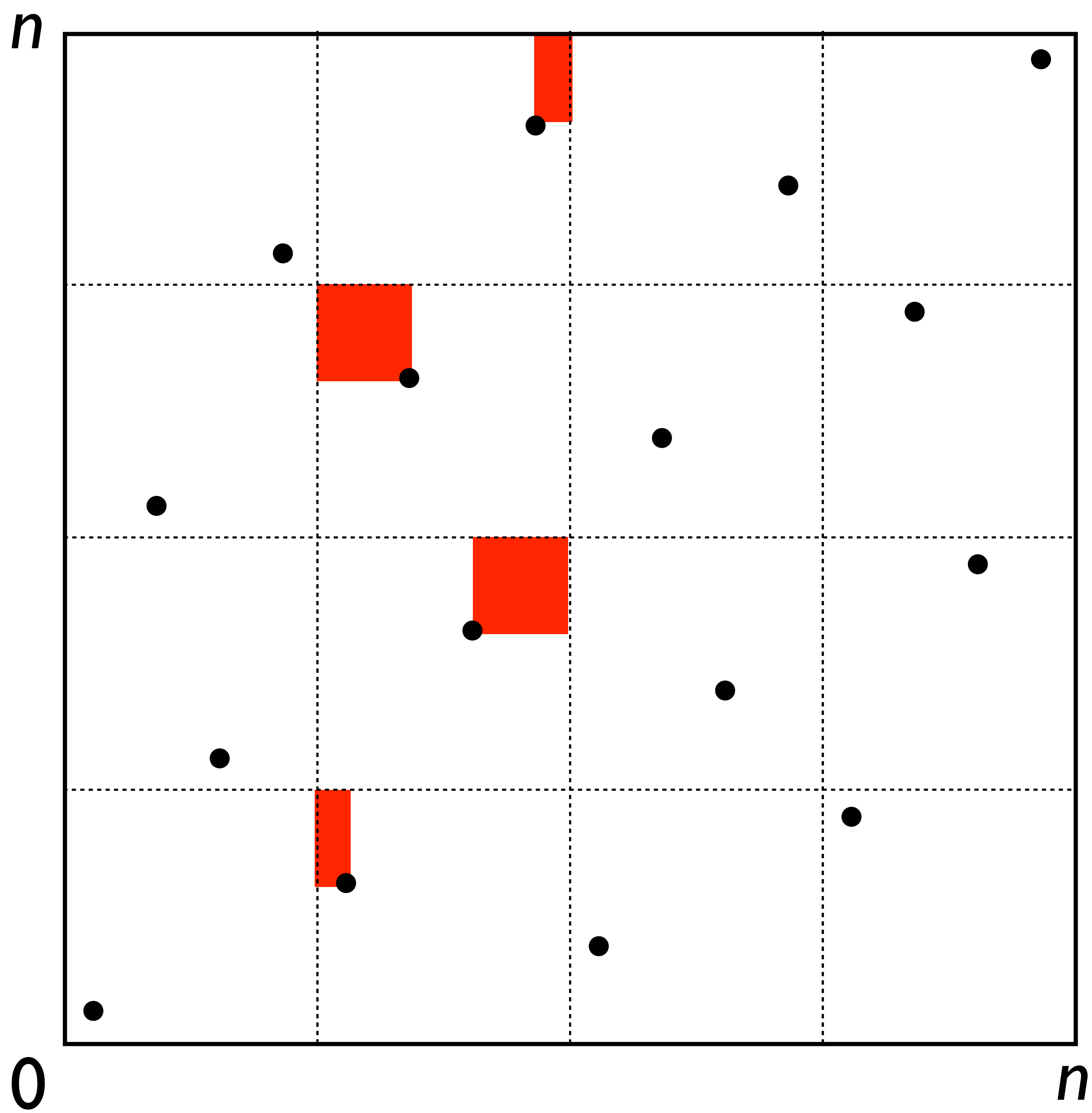
Large Corner Volume



Large Corner Volume

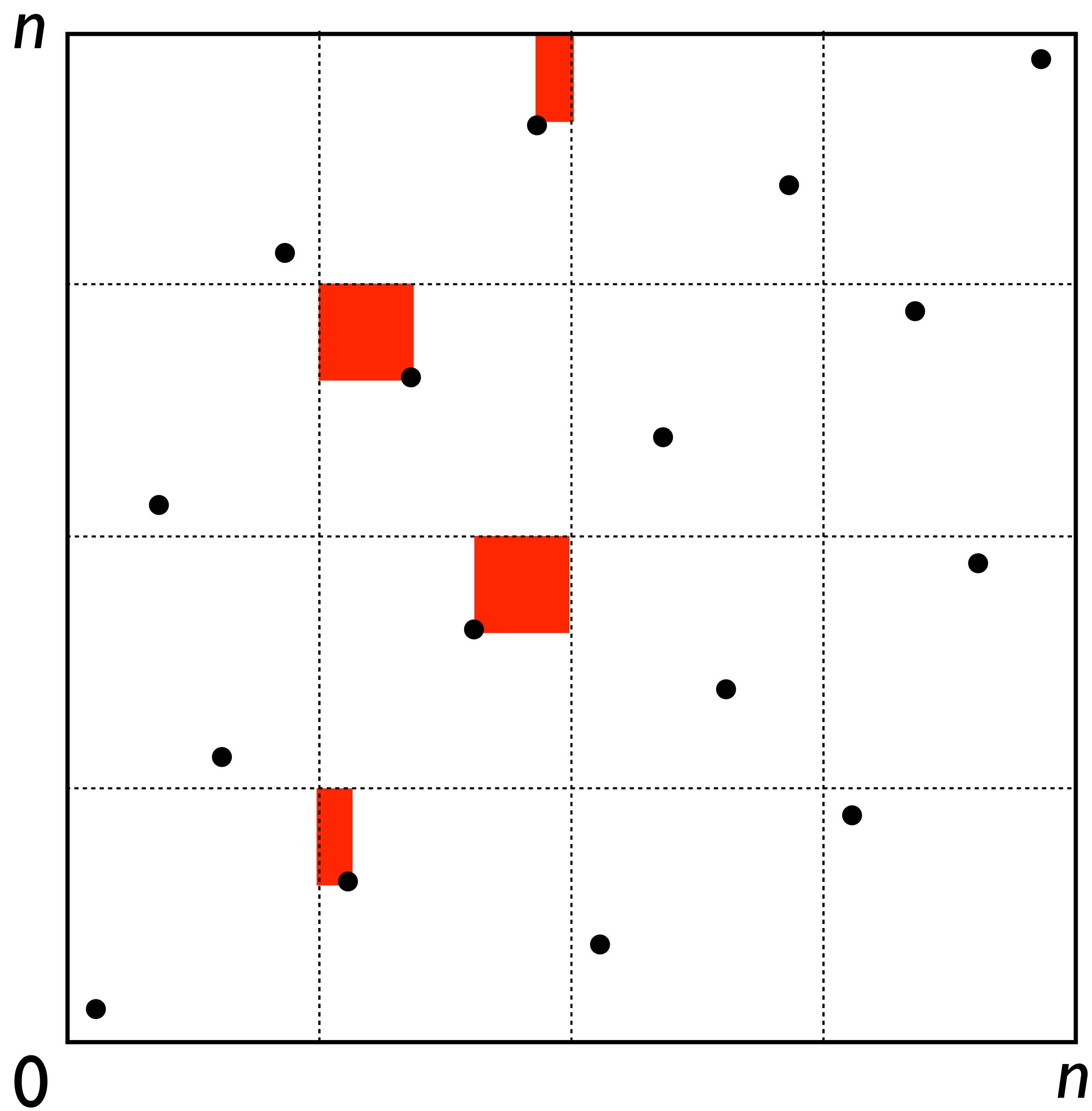


Large Corner Volume



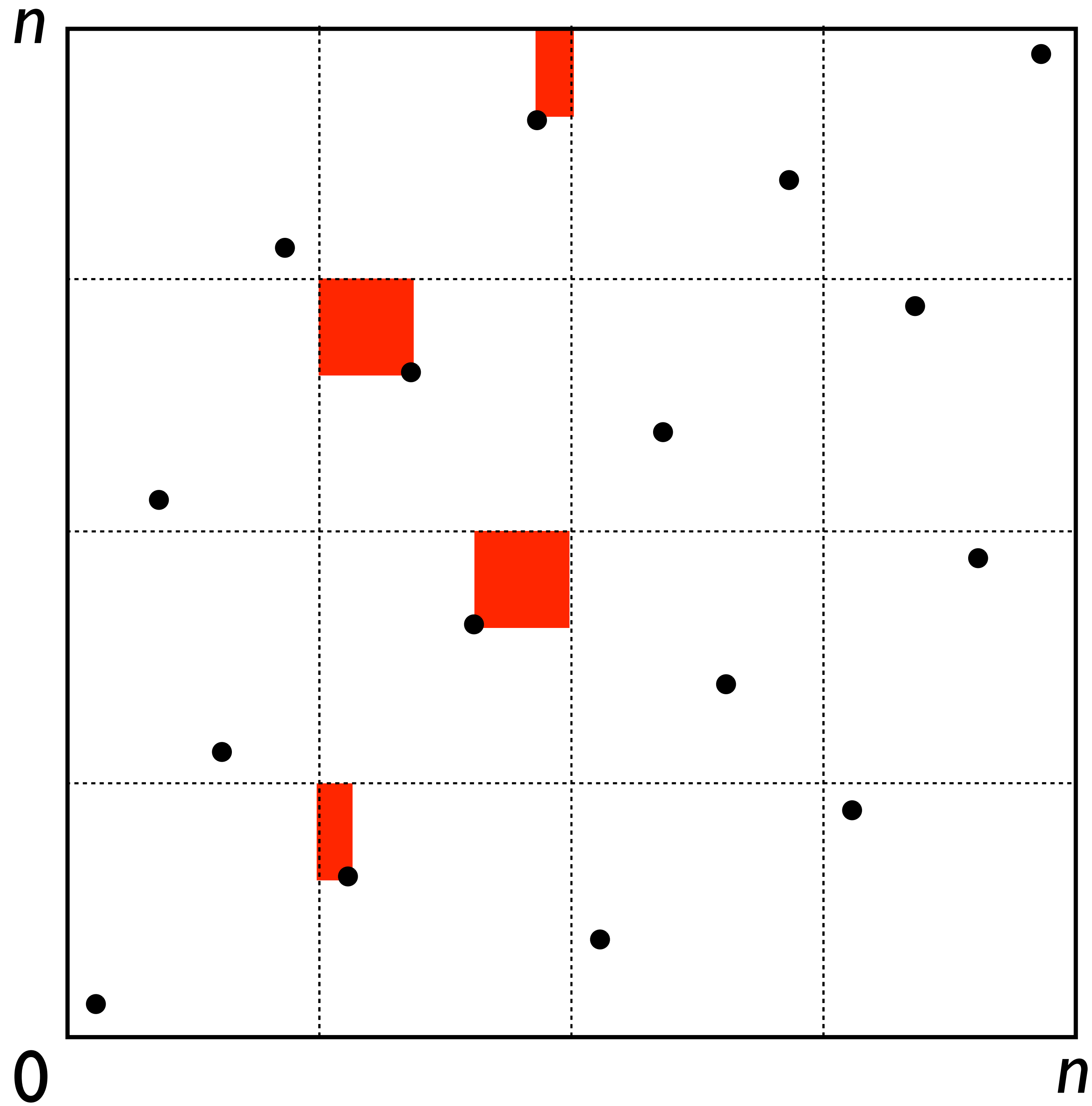
$\frac{1}{4}n$ points with $\geq \frac{1}{8} * x$ length and $\geq \frac{1}{8} * y$ length.

Large Corner Volume



$\frac{1}{4}n$ points with corner volume $\geq \frac{1}{64}n$.

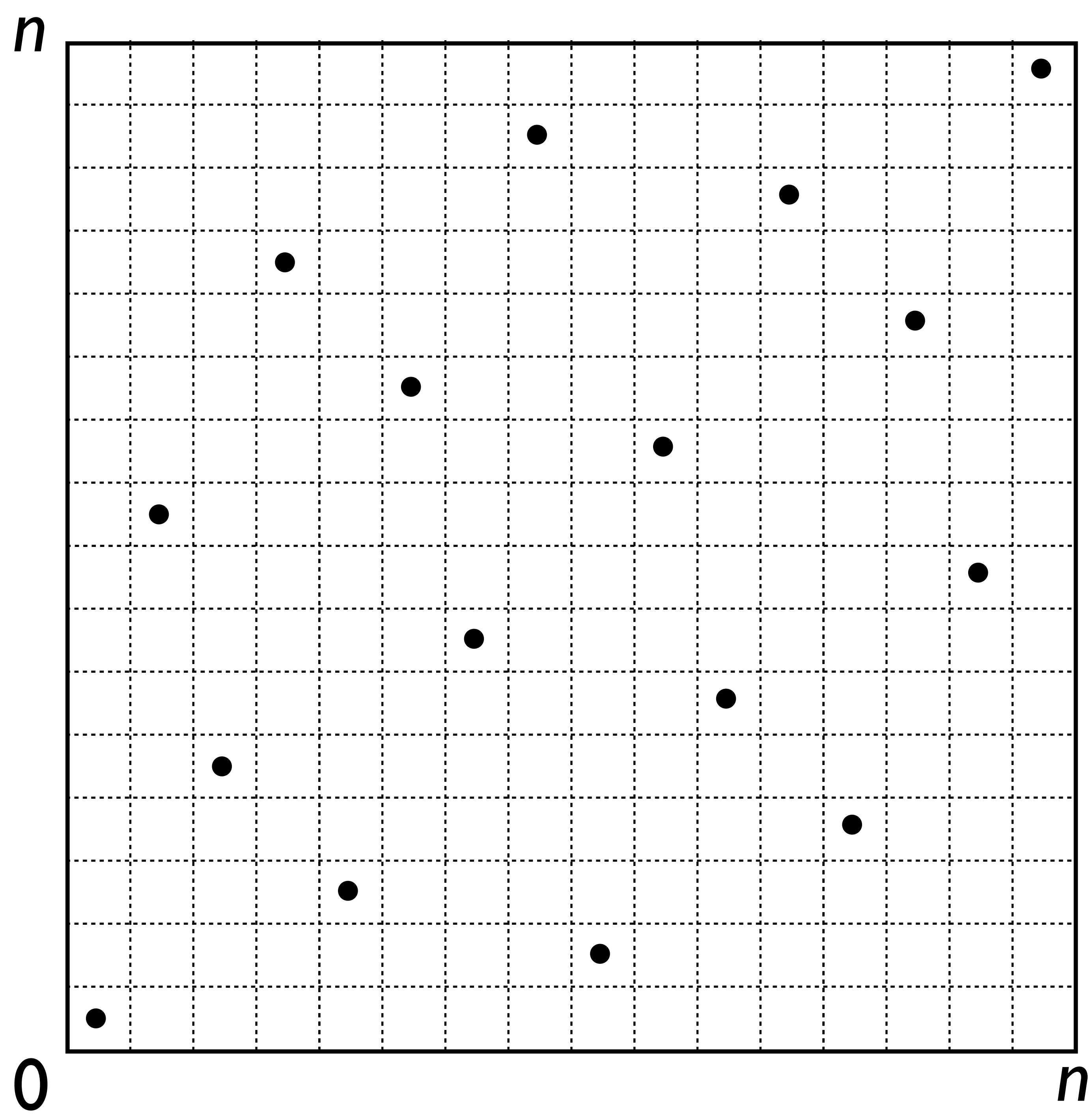
Large Corner Volume



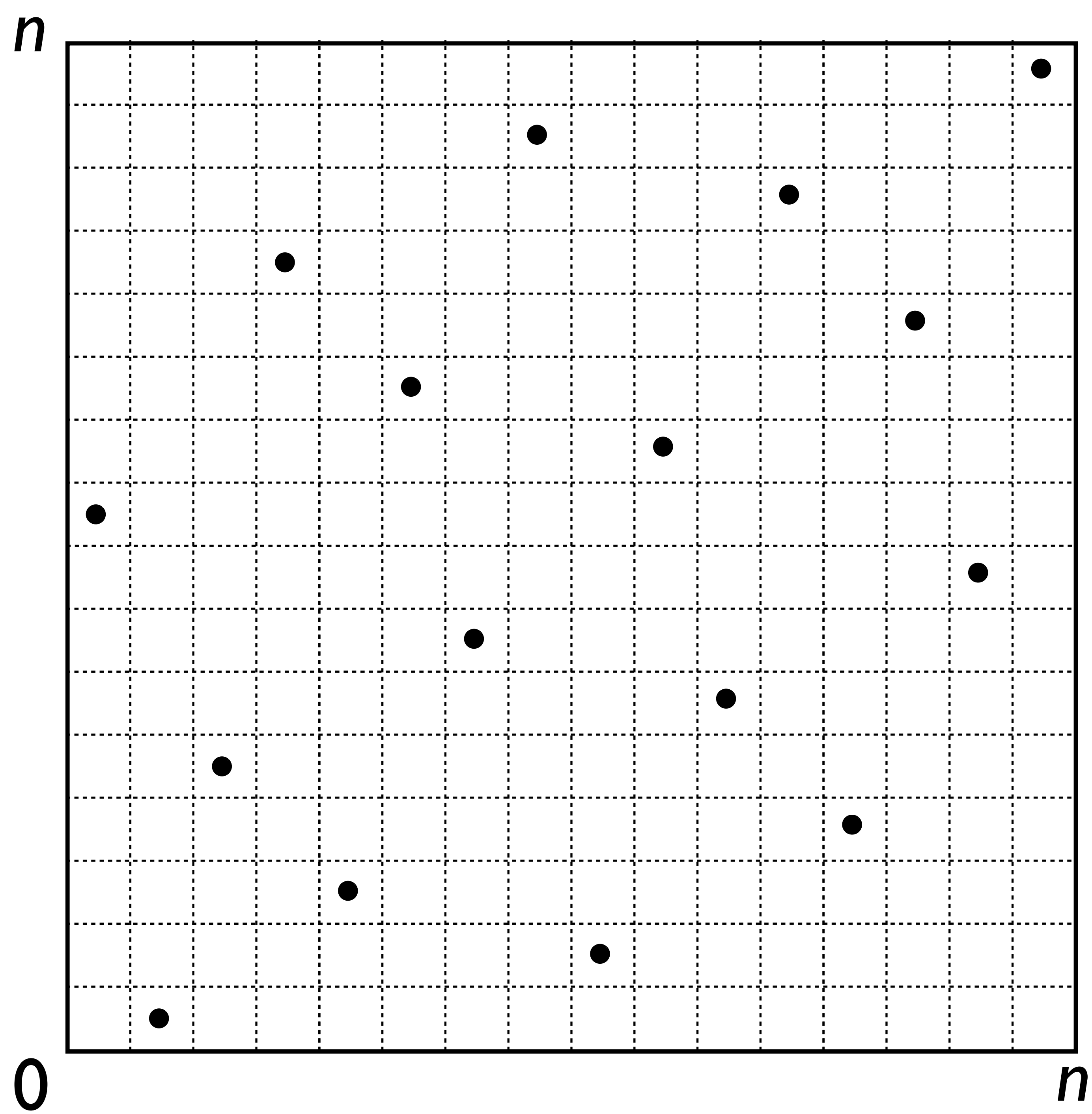
$\log n$ layers $\Rightarrow$ corner volume sum $\geq cn^2 \log n$.

Goal: Collection $\mathcal{P}^*$ of $2^{\Omega(n \log n)}$ point sets, s.t. $\forall$ $P_1, P_2 \in \mathcal{P}^*$, we have $\mathrm{disc}(P_1 \cup P_2, \mathcal{R}_2) \geq c \log n$.

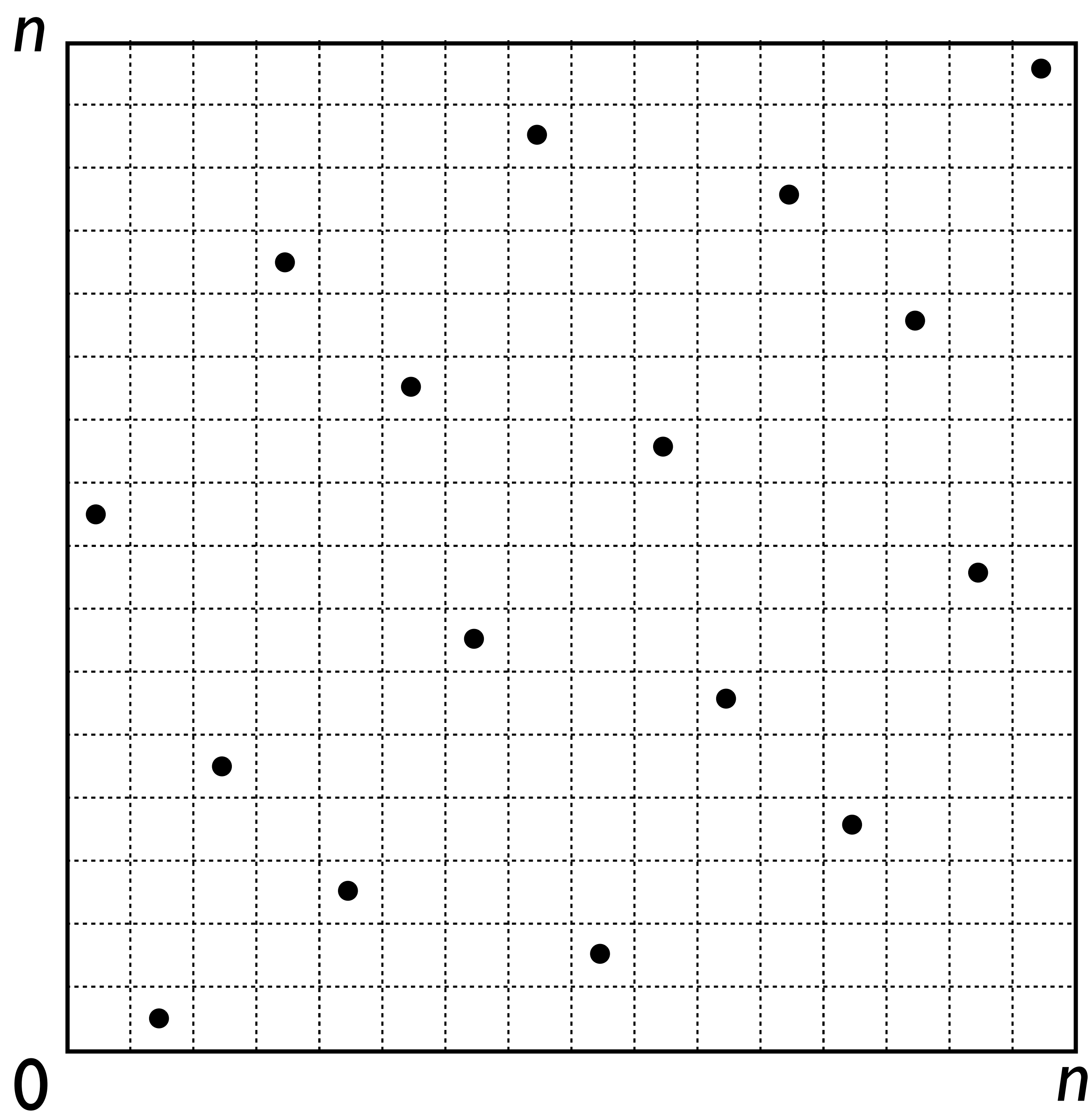
CD of Union of 2 Binary Sets



CD of Union of 2 Binary Sets

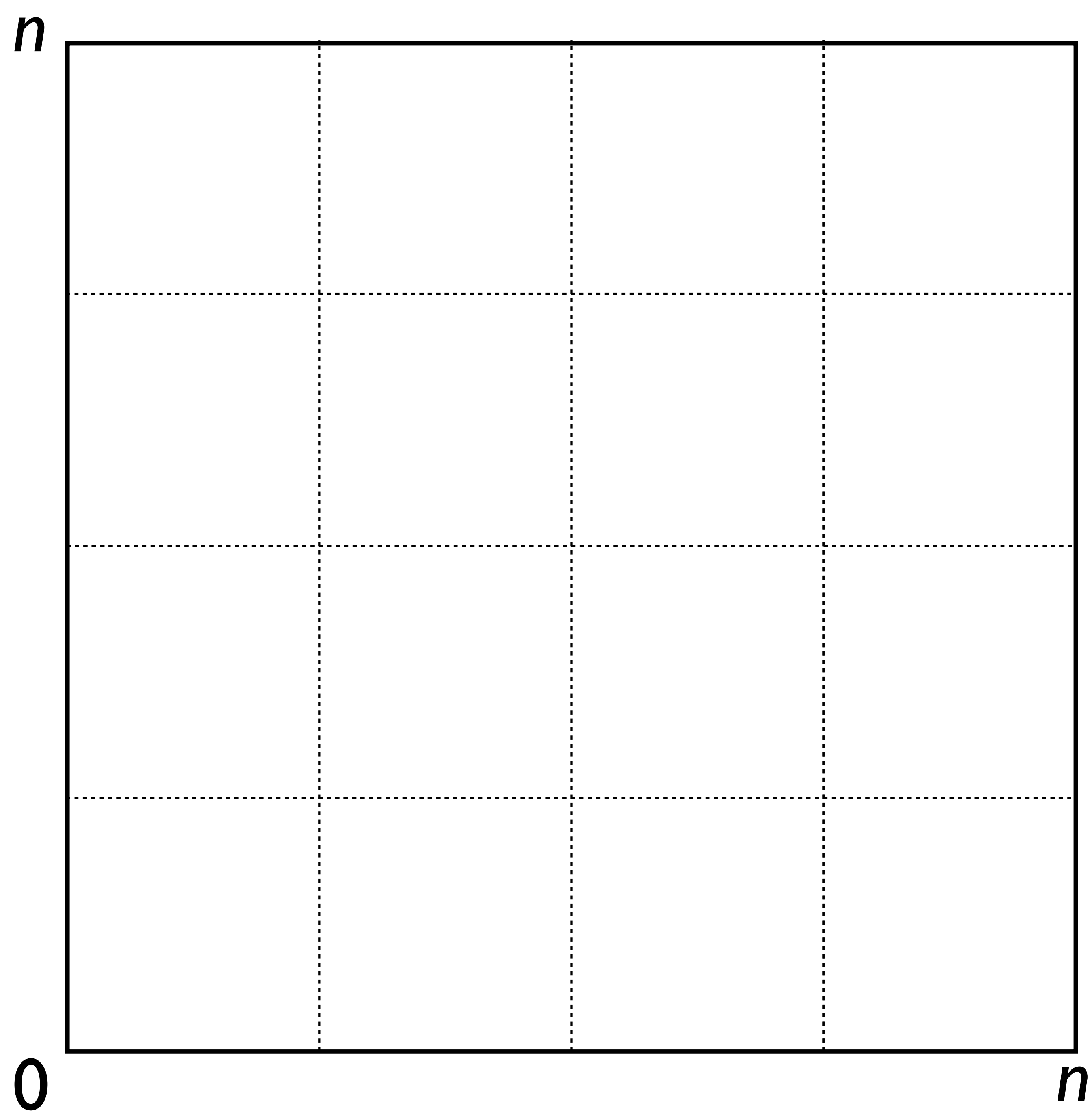


CD of Union of 2 Binary Sets

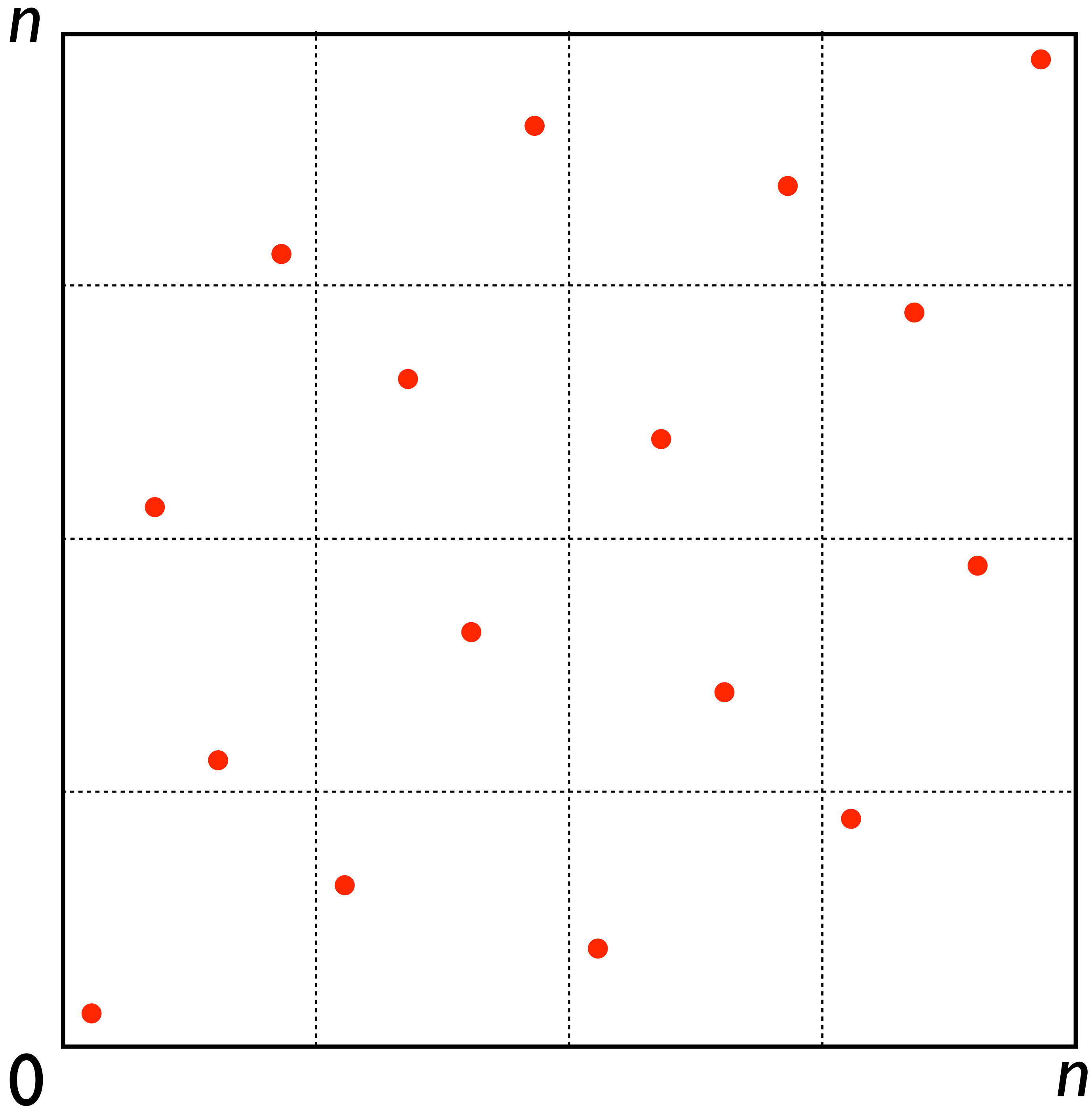


- Need to refine the point sets!

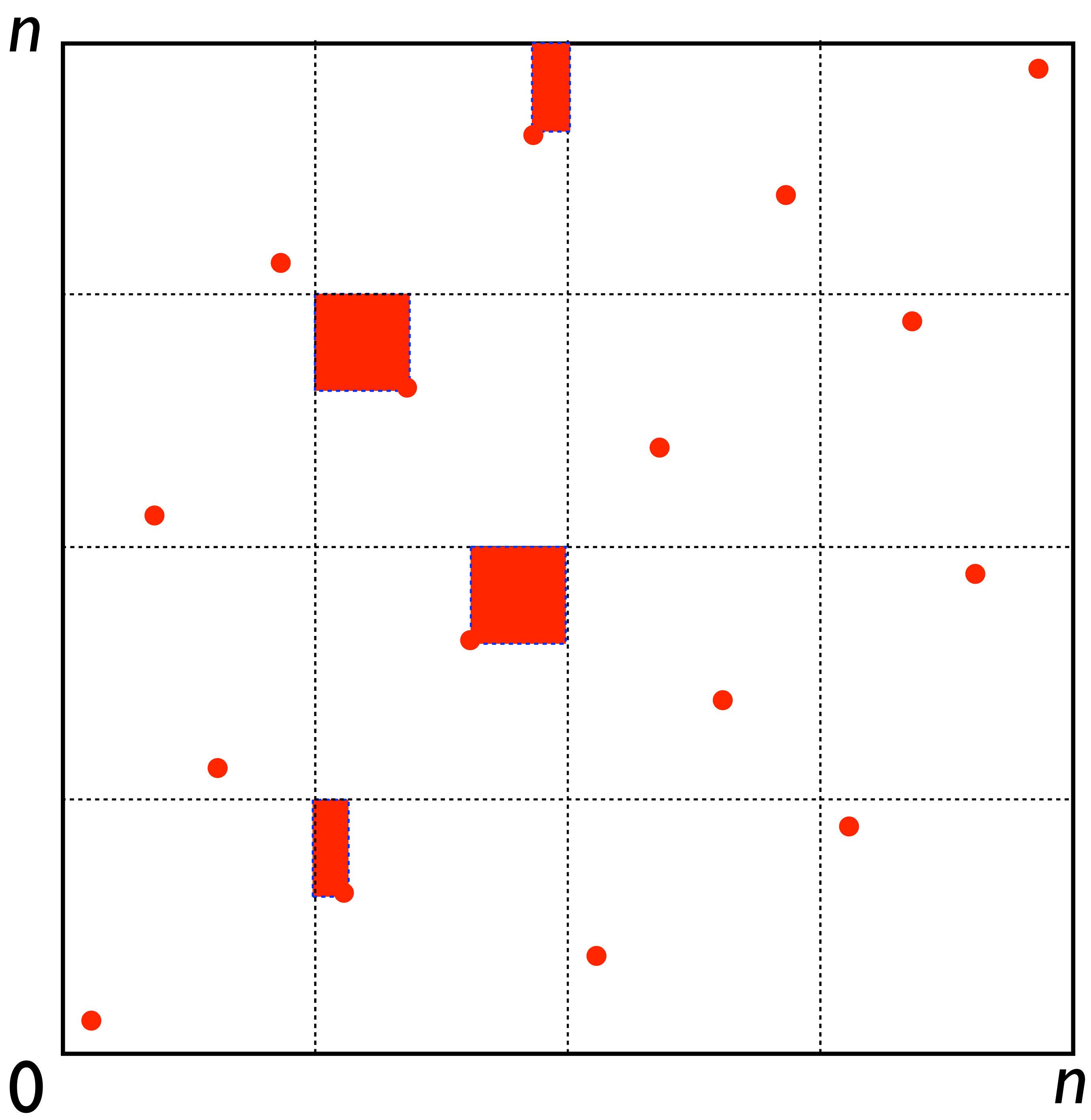
Corner Volume Distance



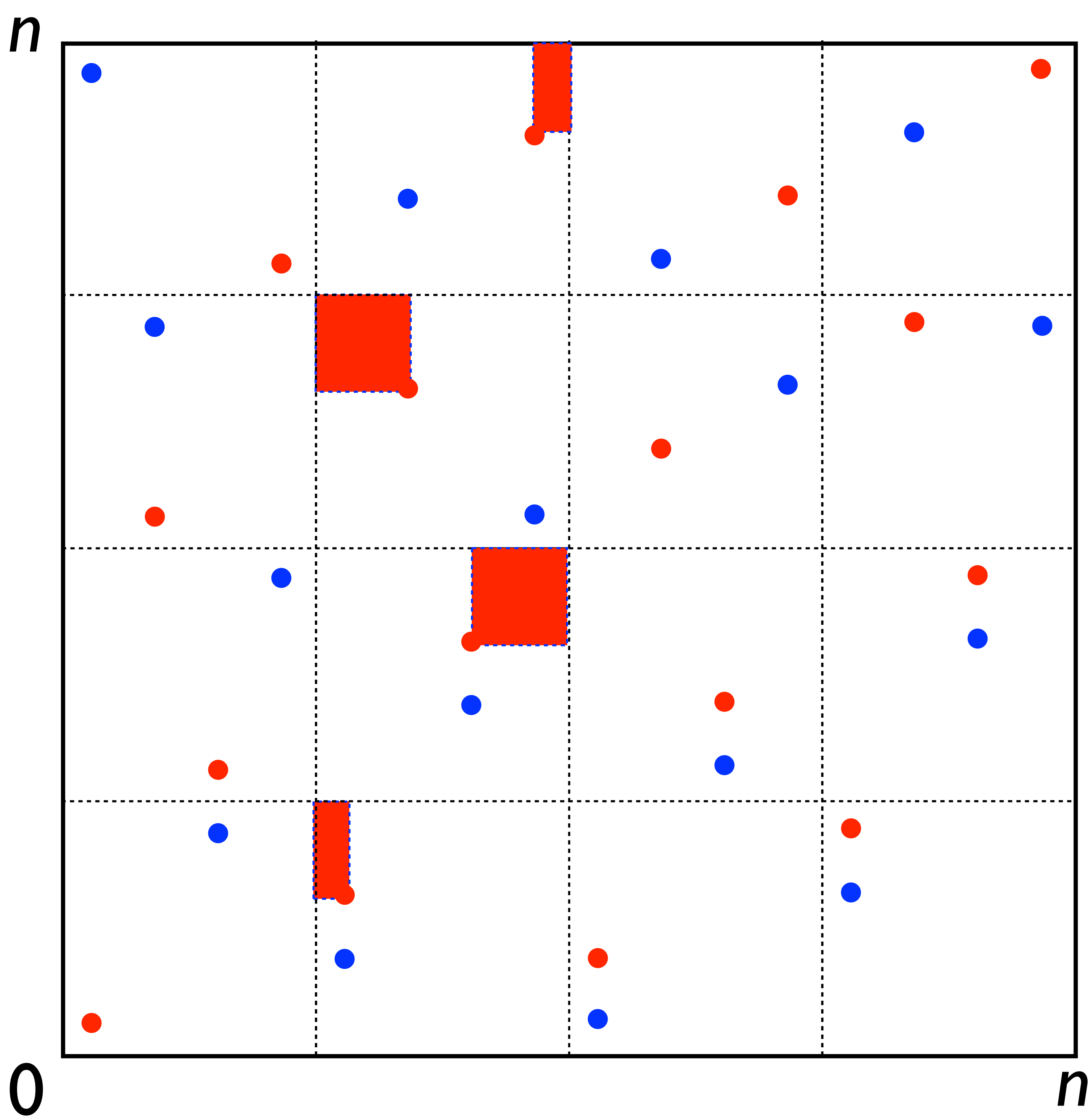
Corner Volume Distance



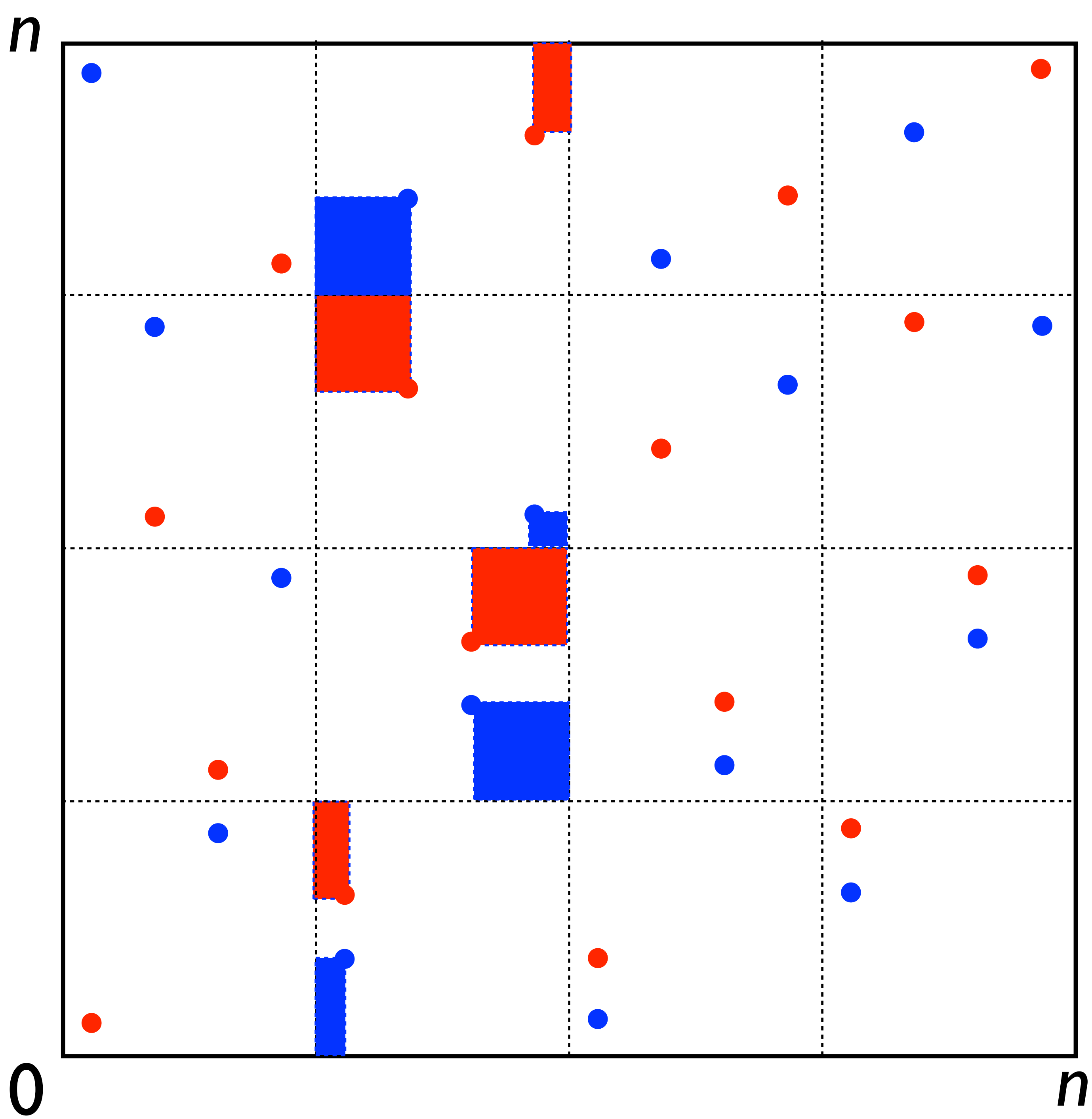
Corner Volume Distance



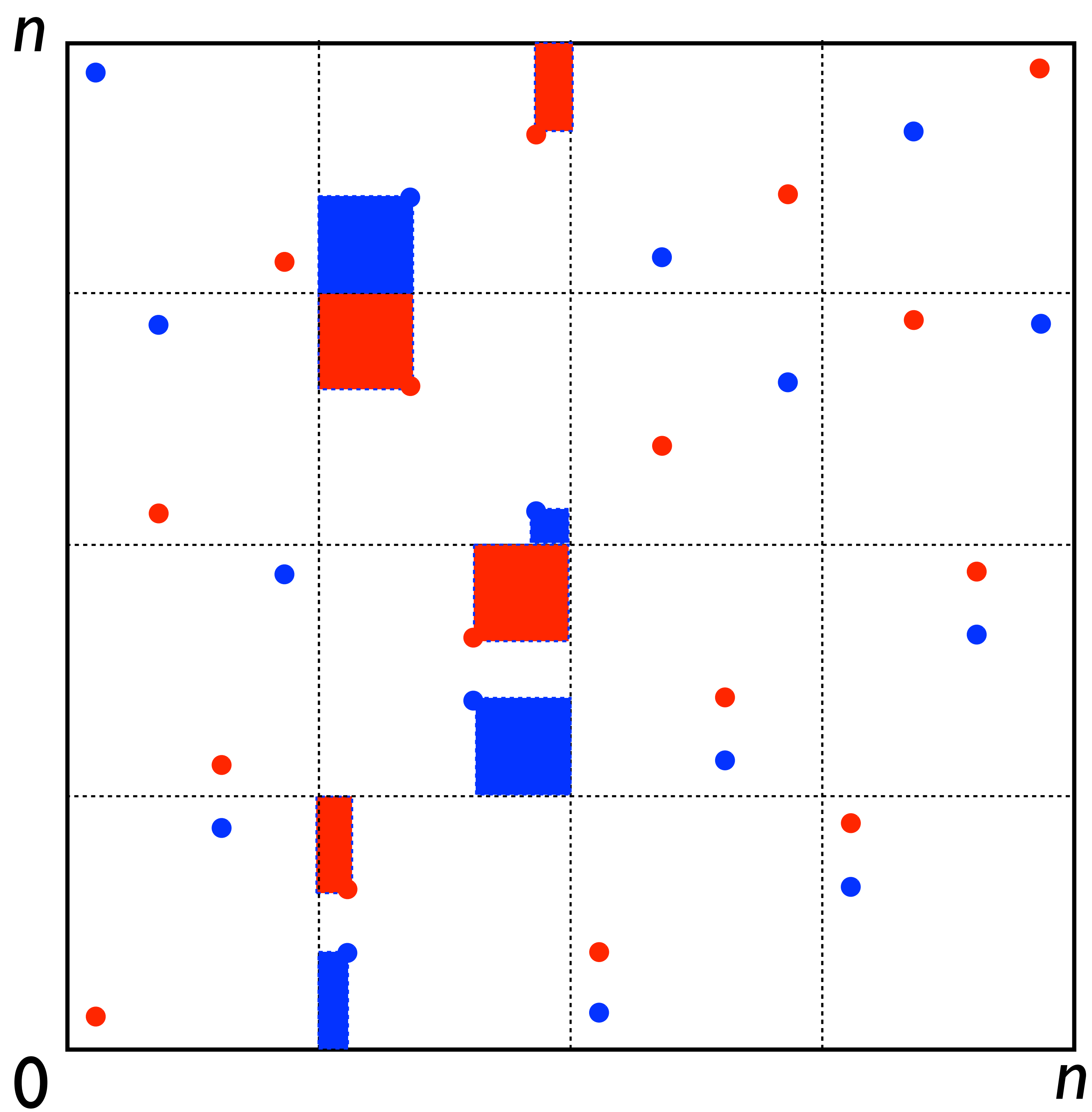
Corner Volume Distance



Corner Volume Distance



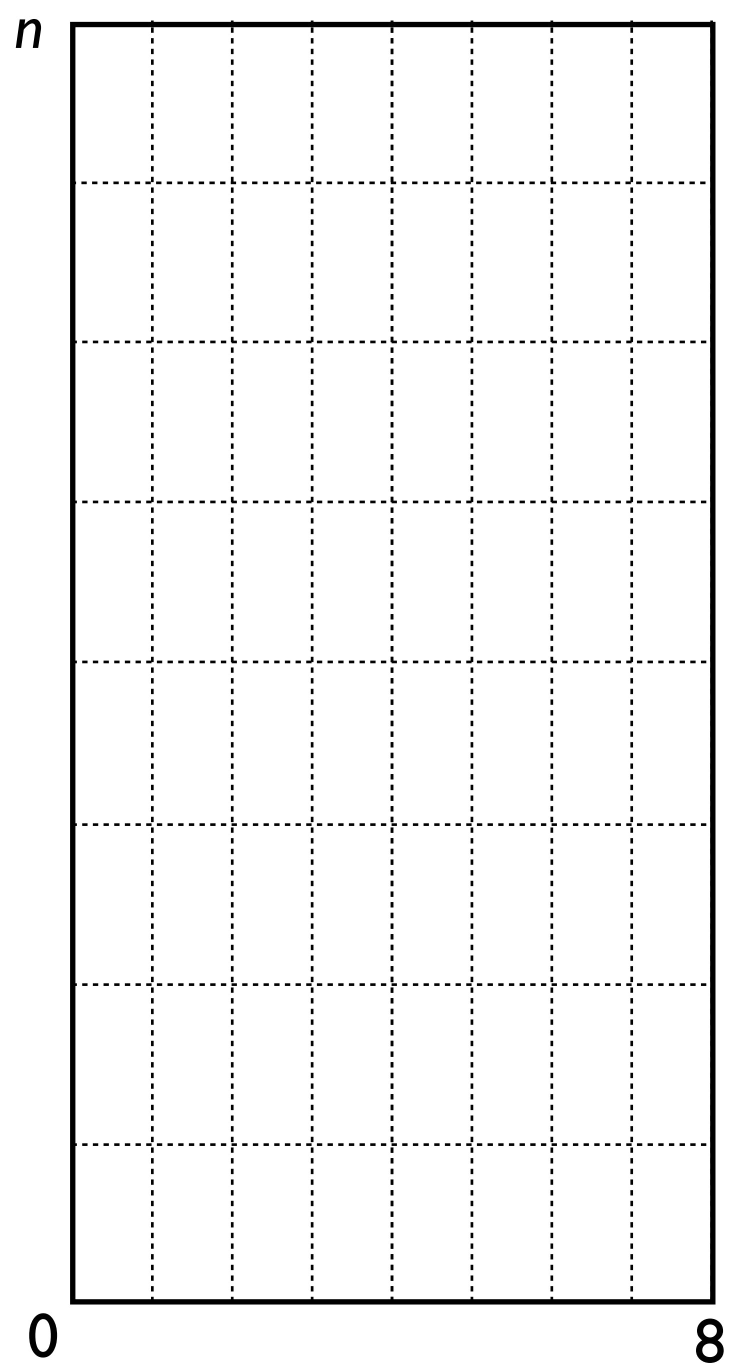
Corner Volume Distance



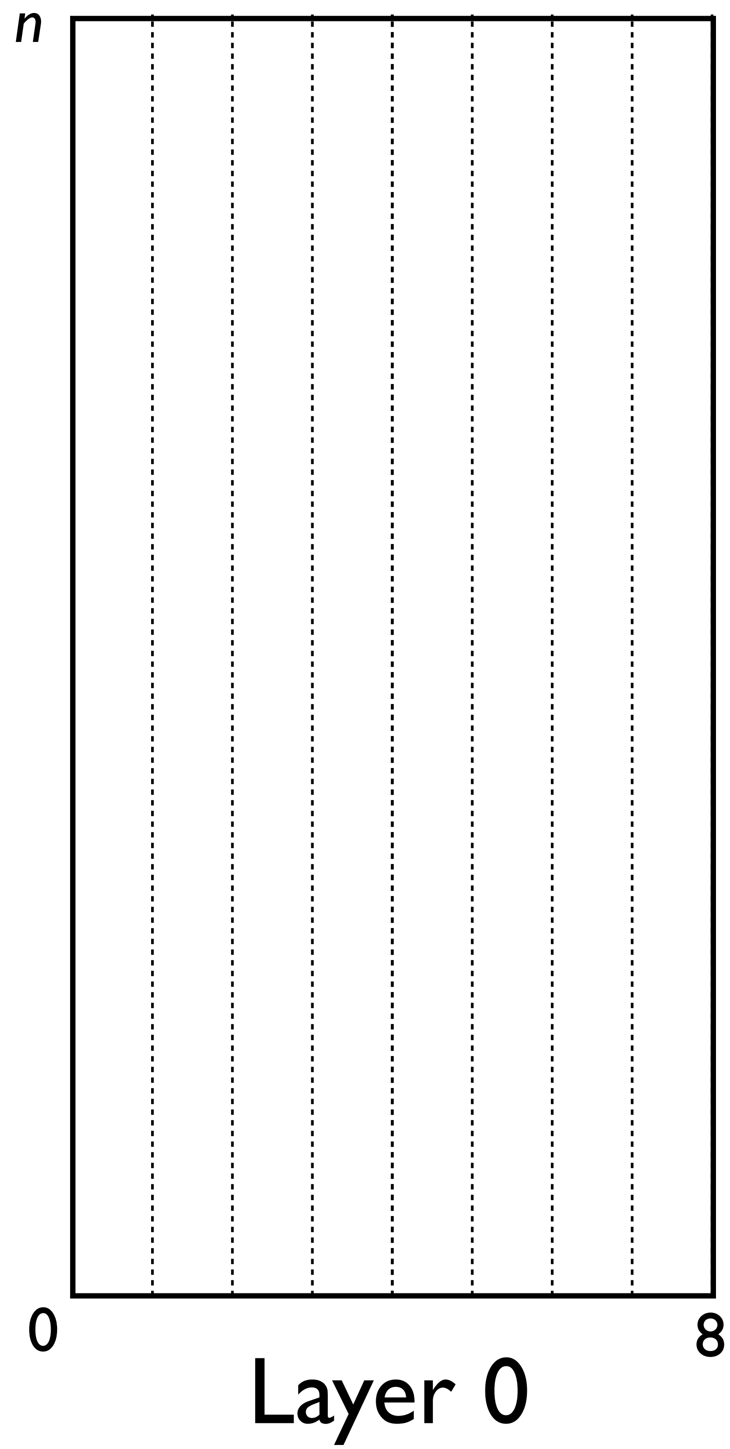
Lemma: If corner volume distance $\geq cn^2 \log n$, then $\mathrm{disc}(P_1 \cup P_2) = \Omega(\log n)$.

Goal: Find a subcollection $\mathcal{P}^*$ of $2^{\Omega(n \log n)}$ point sets, s.t. $\forall\, P_1, P_2 \in \mathcal{P}^*$, the corner volume distance $\Delta(P_1, P_2) \geq cn^2 \log n$.

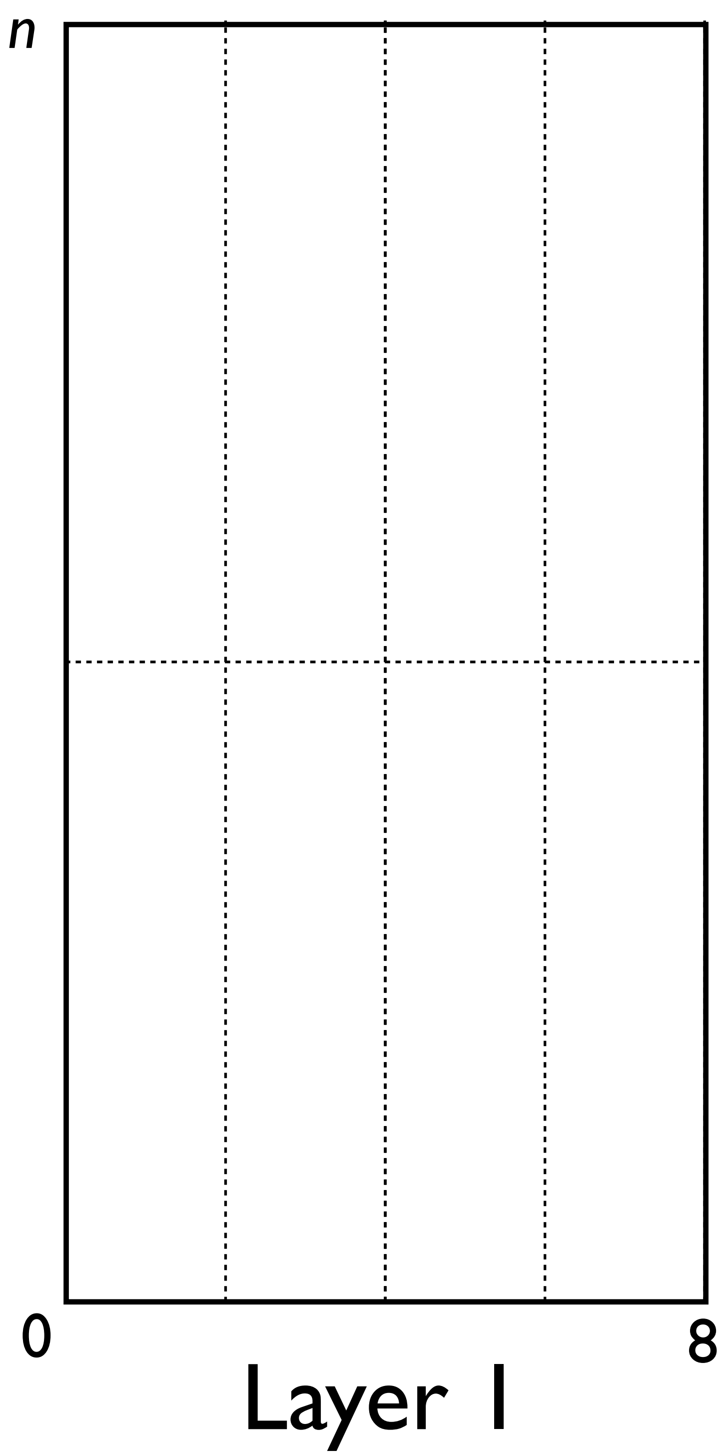
Binary Nets with Large CVD



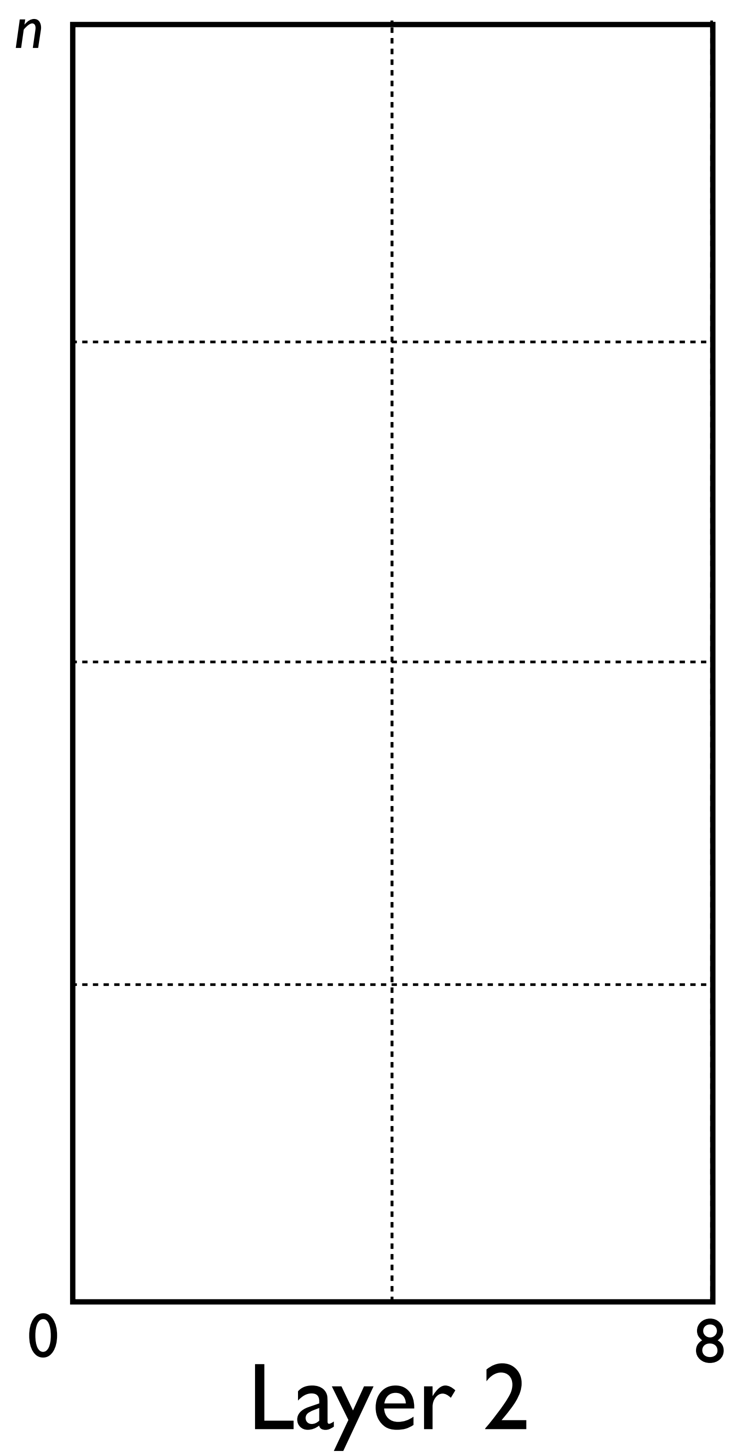
Binary Nets with Large CVD



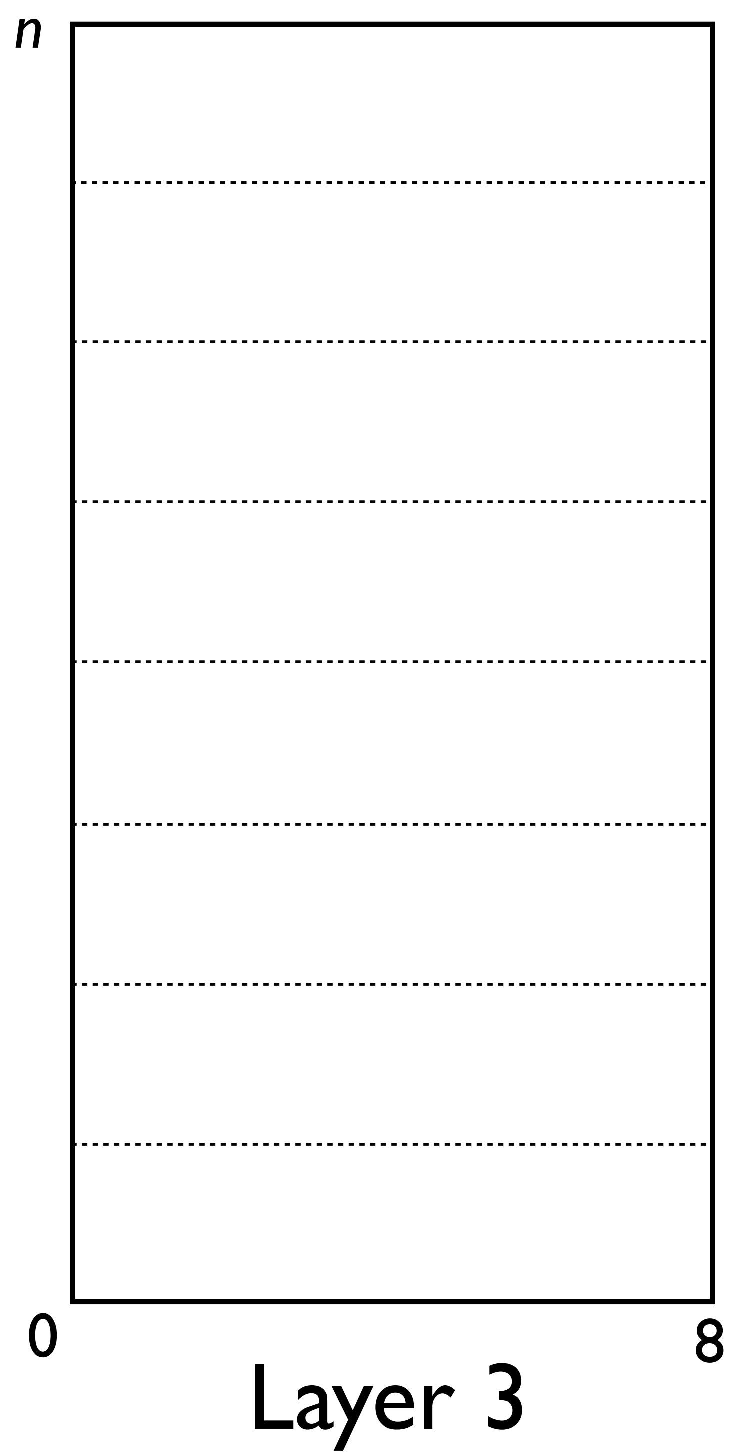
Binary Nets with Large CVD



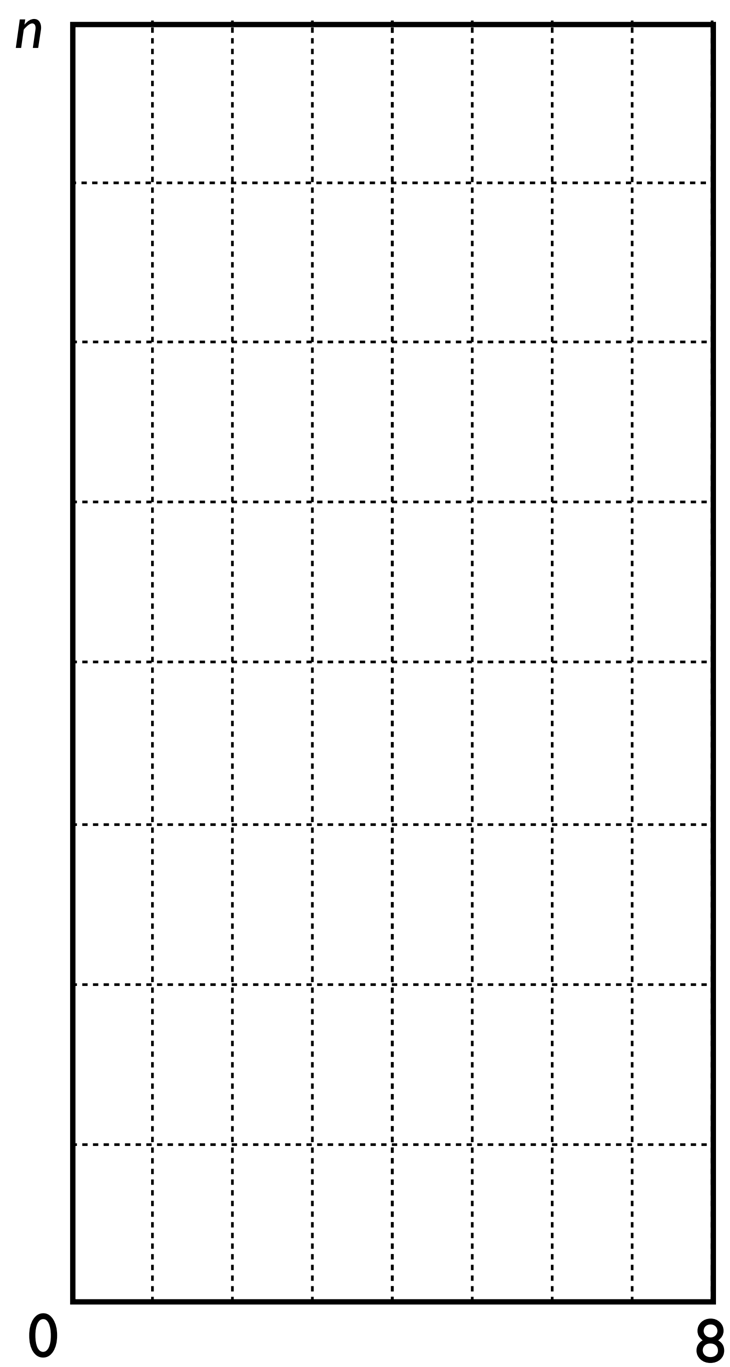
Binary Nets with Large CVD



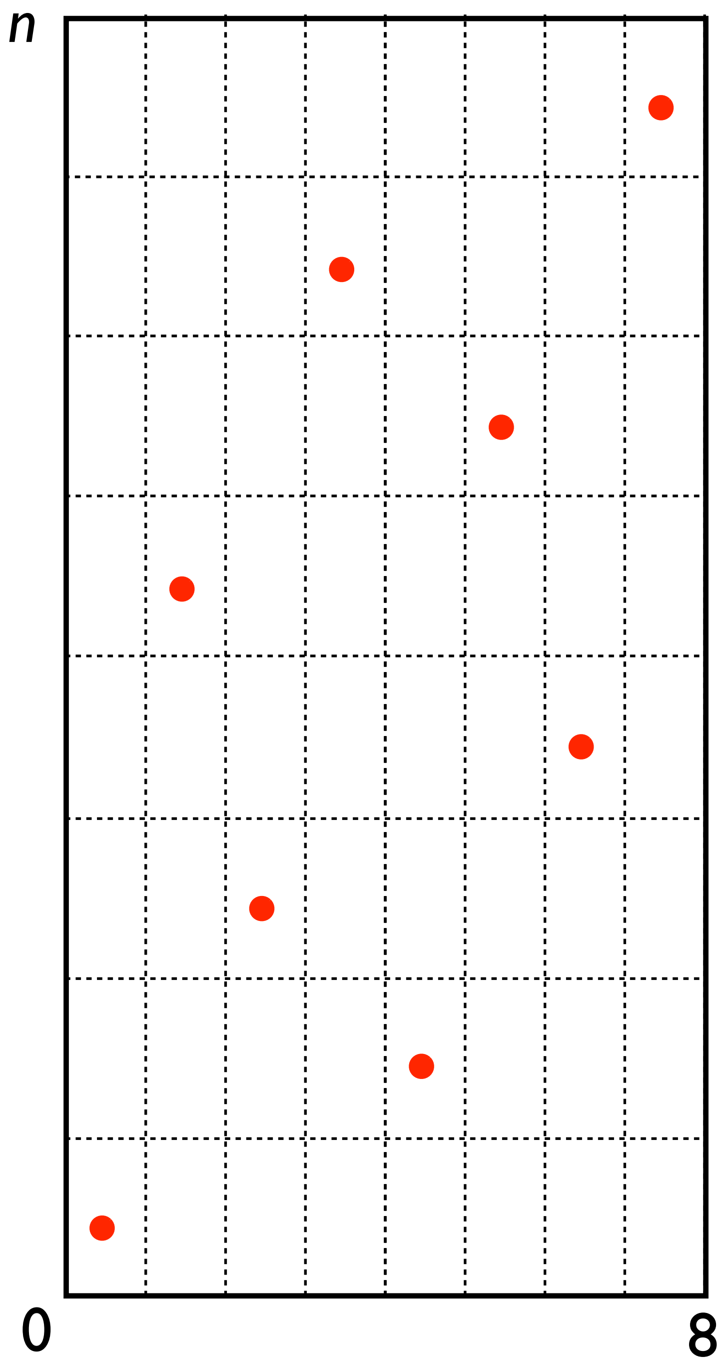
Binary Nets with Large CVD



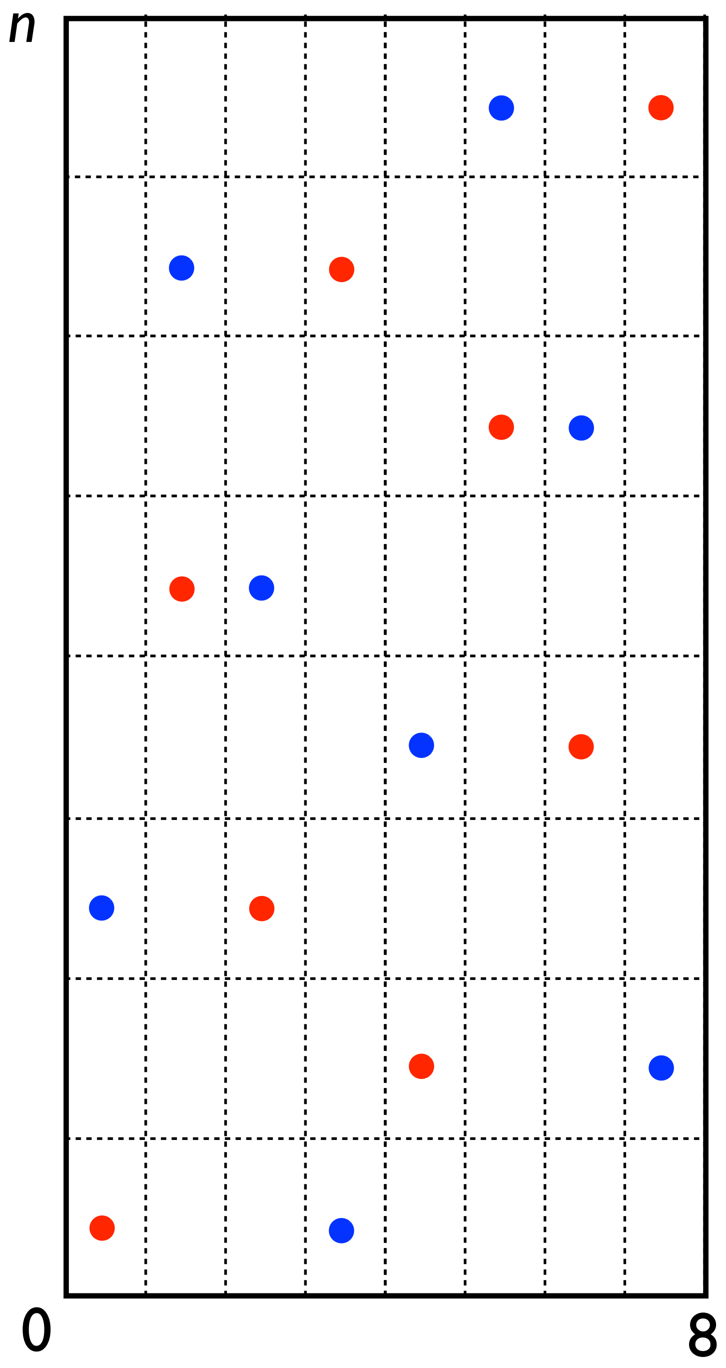
Binary Nets with Large CVD



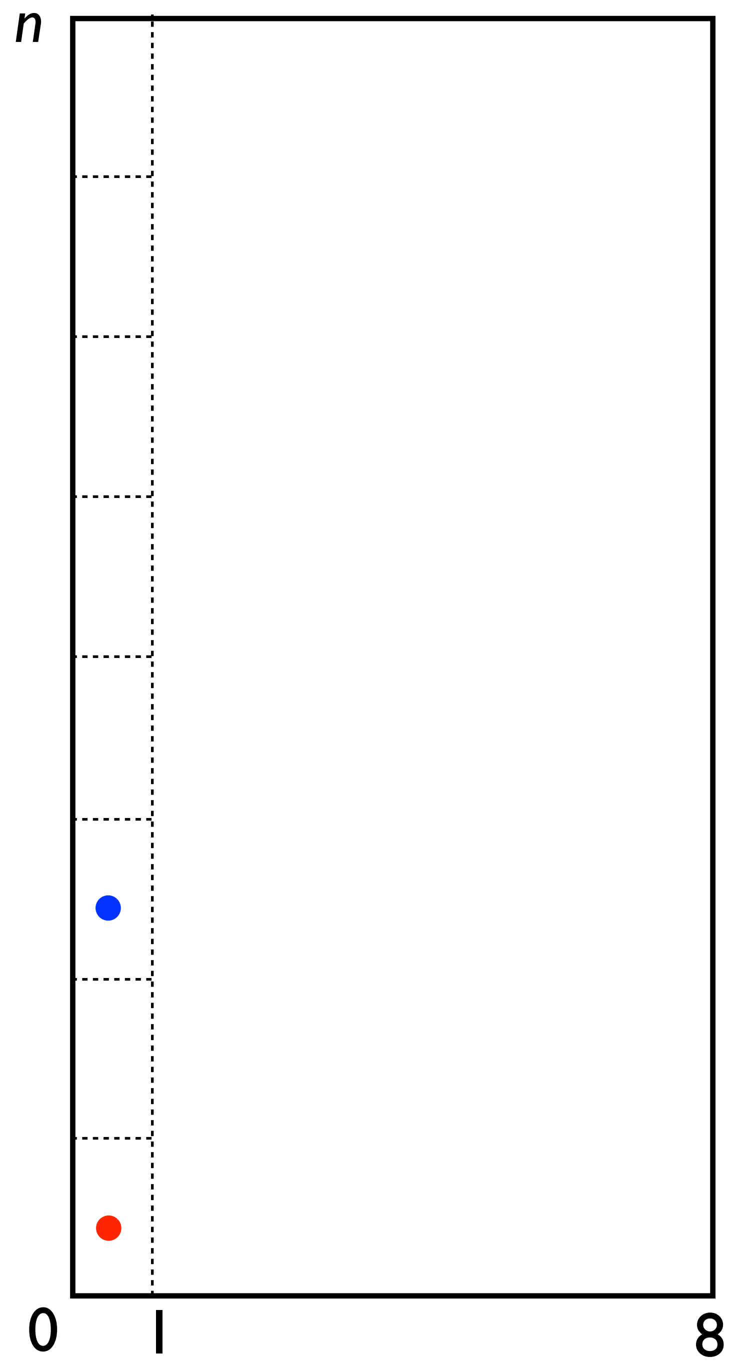
Binary Nets with Large CVD



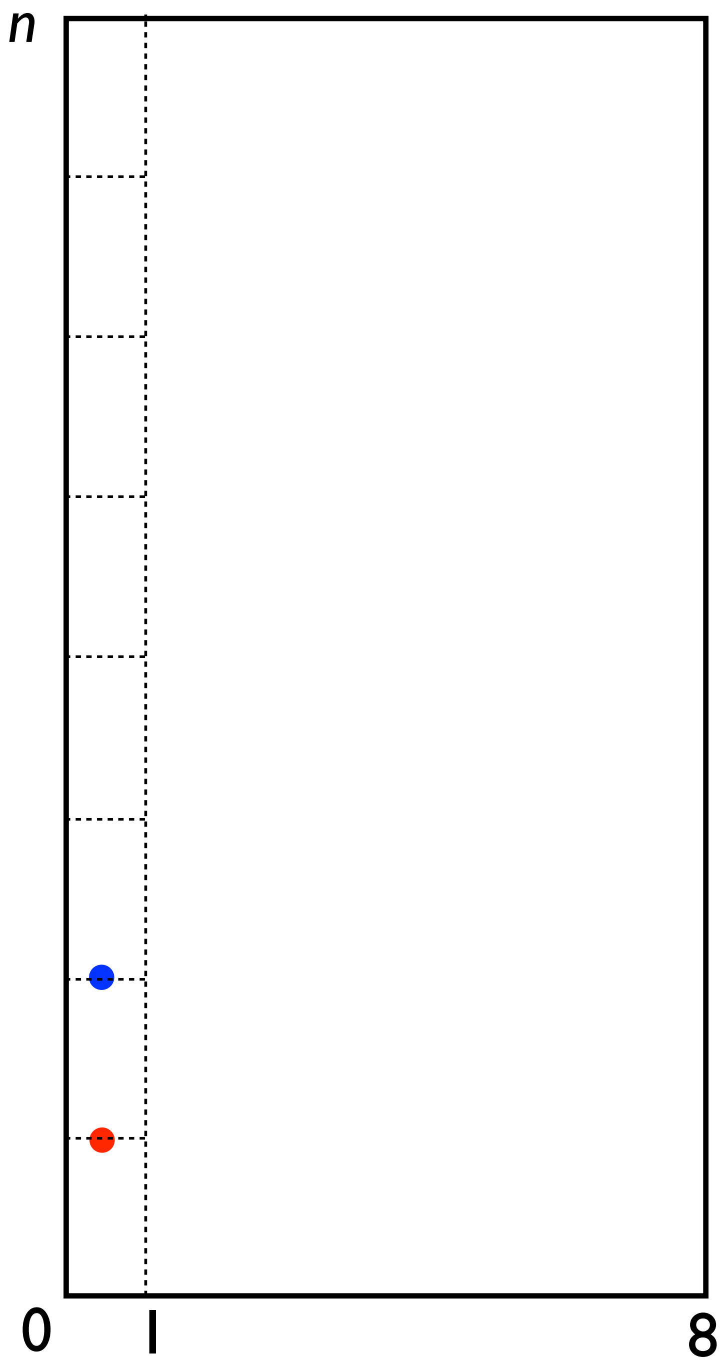
Binary Nets with Large CVD



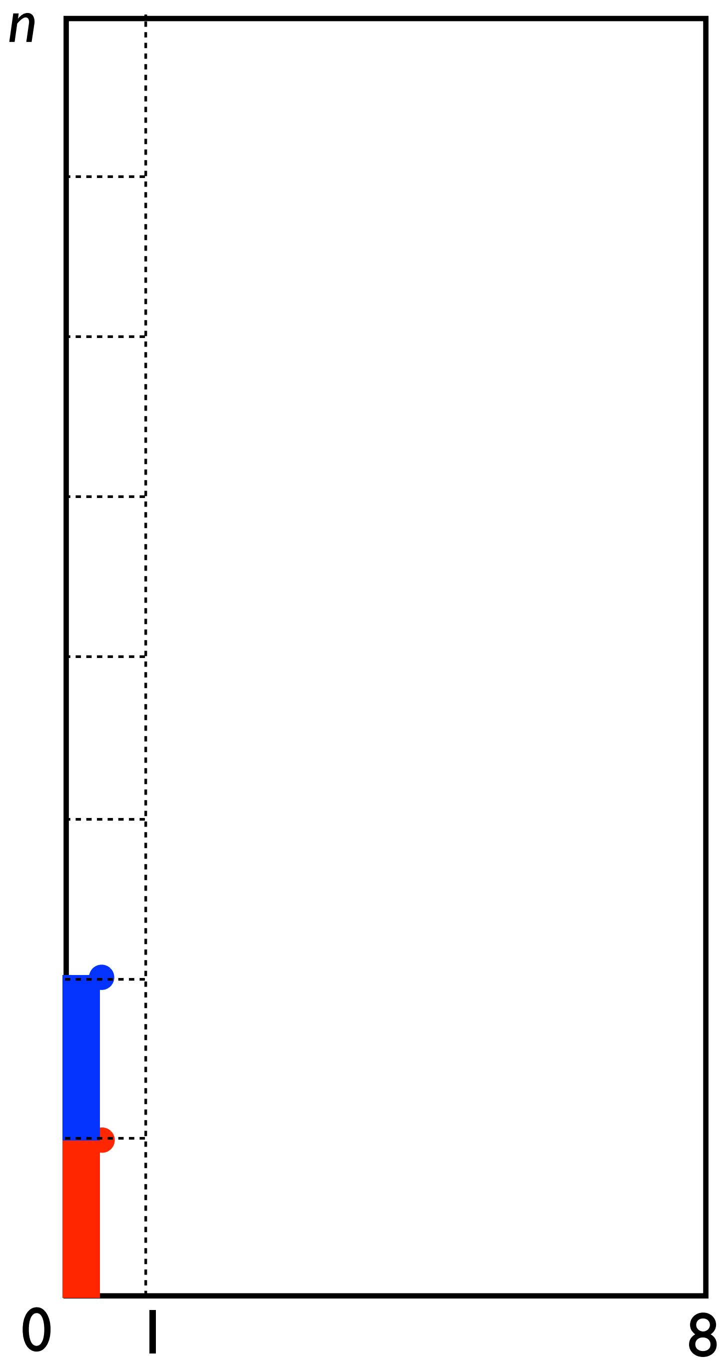
Binary Nets with Large CVD



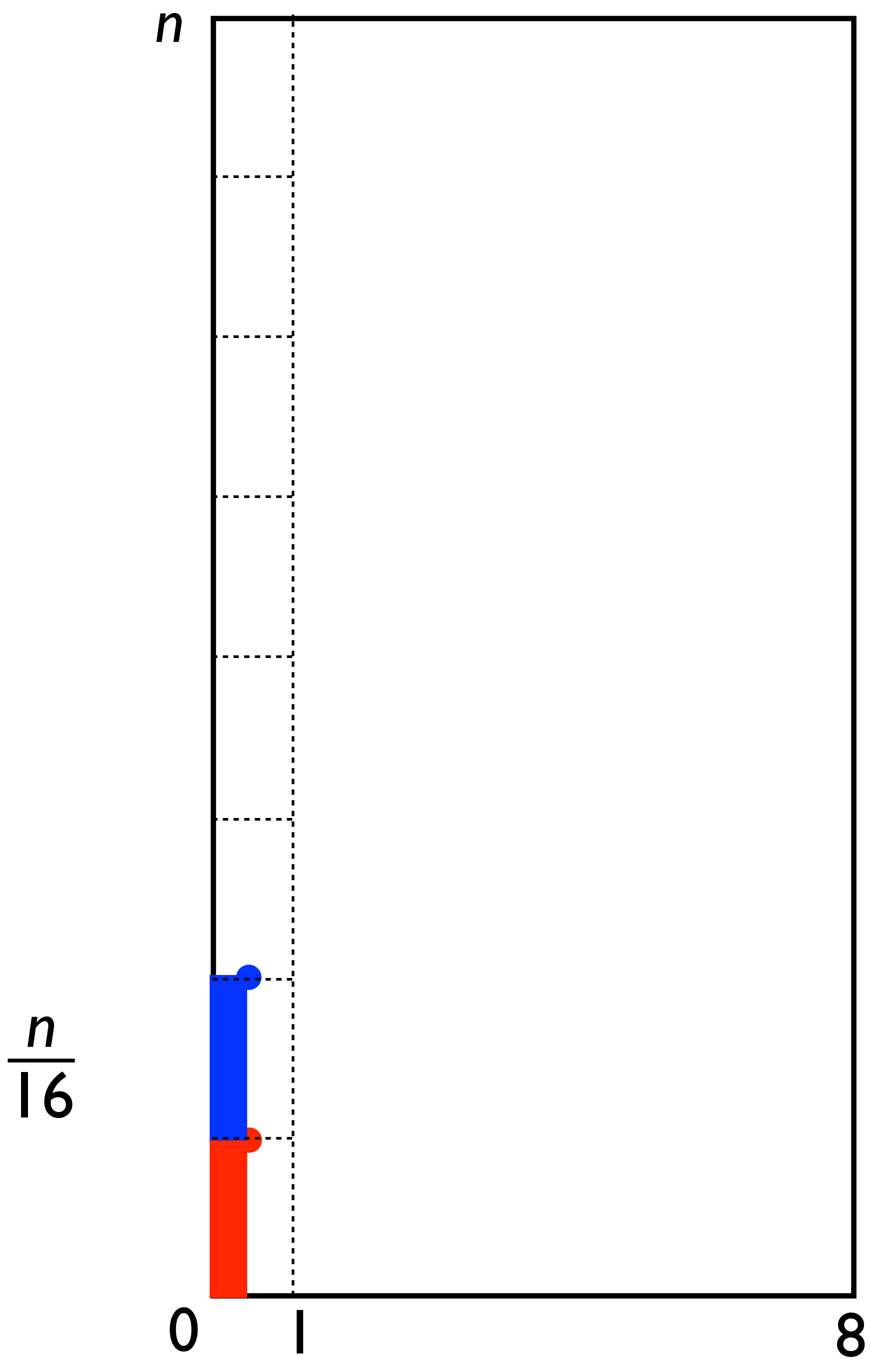
Binary Nets with Large CVD



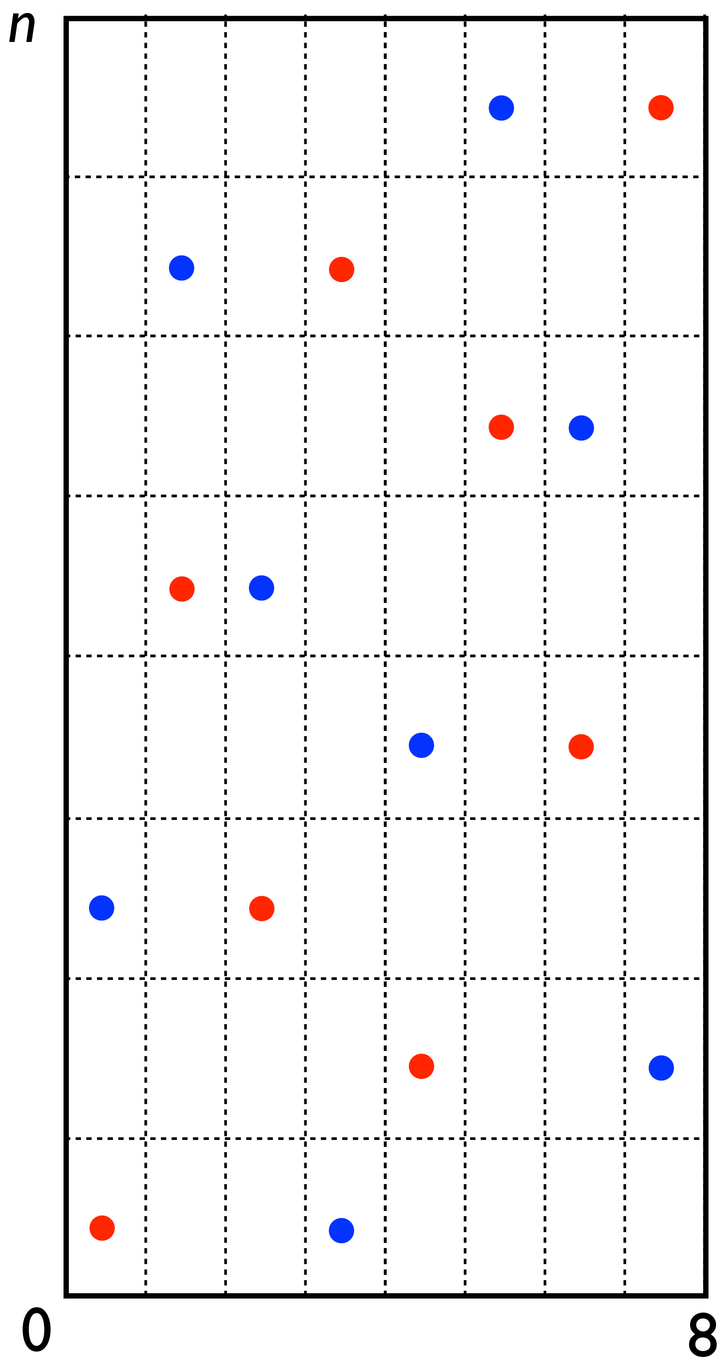
Binary Nets with Large CVD



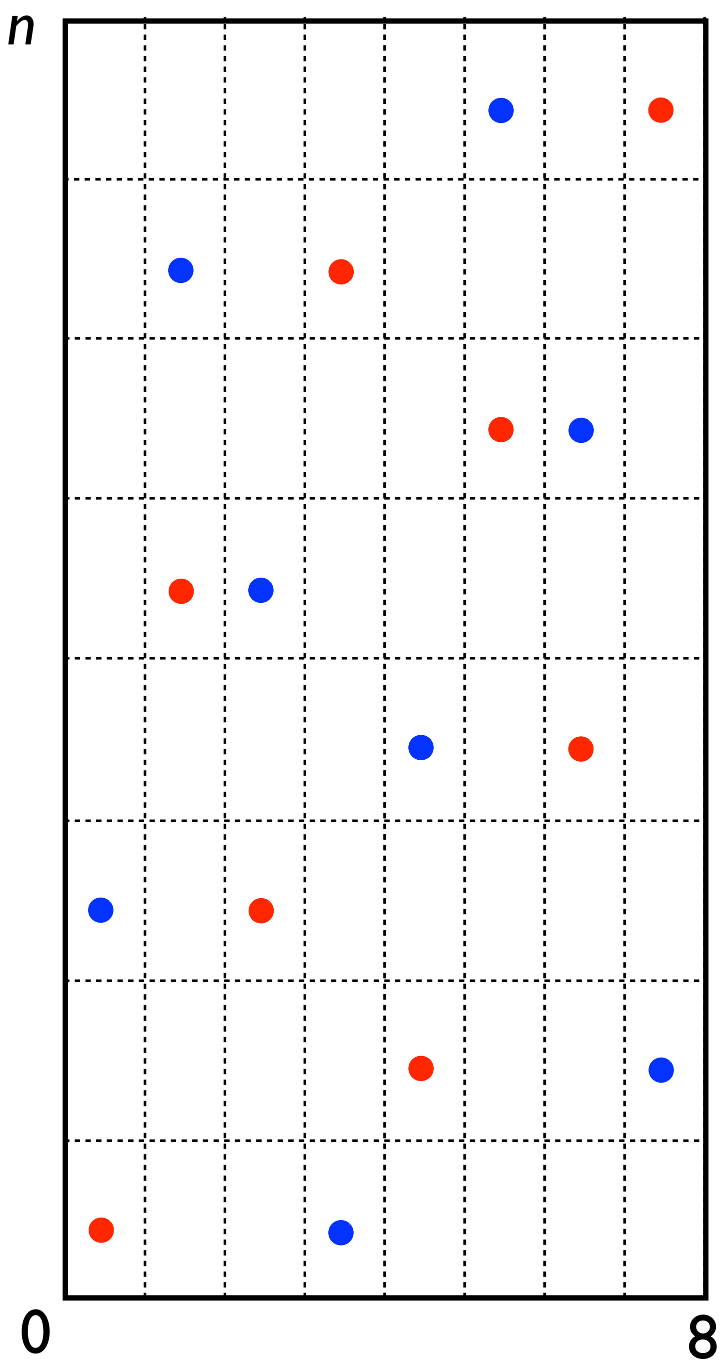
Binary Nets with Large CVD



Binary Nets with Large CVD

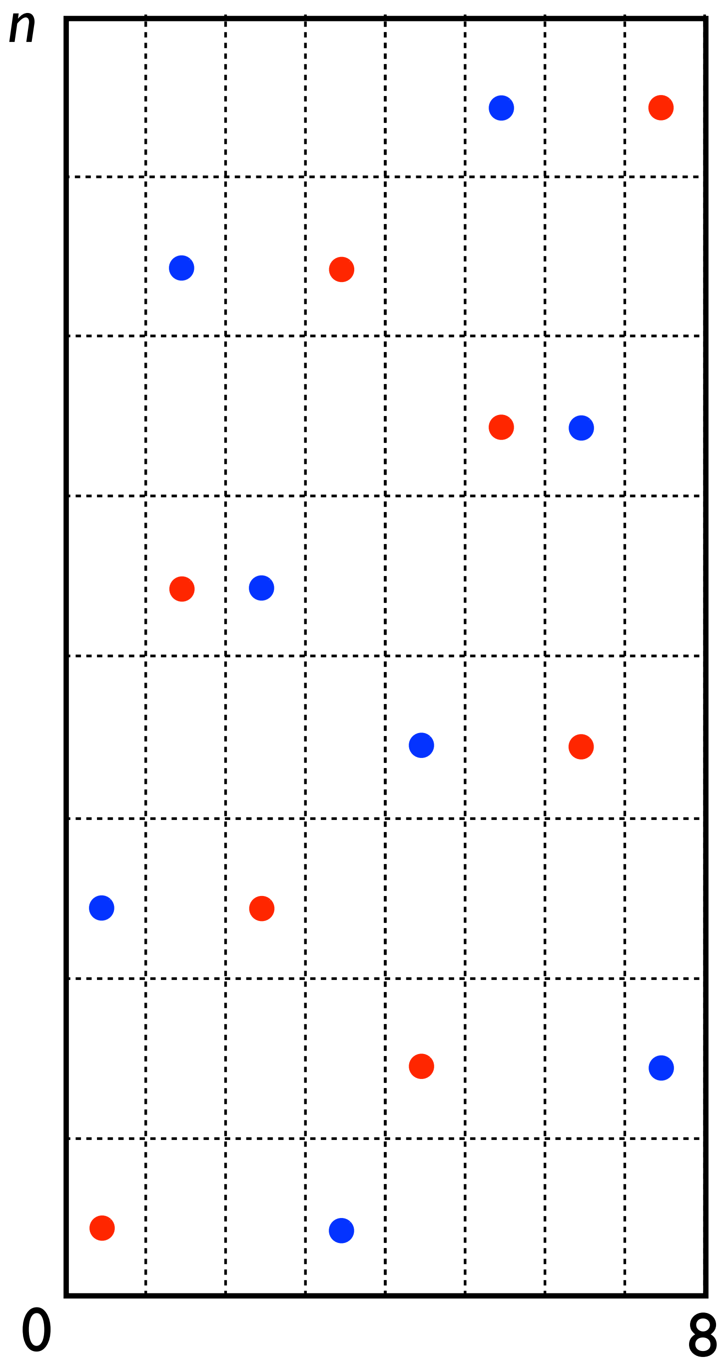


Binary Nets with Large CVD

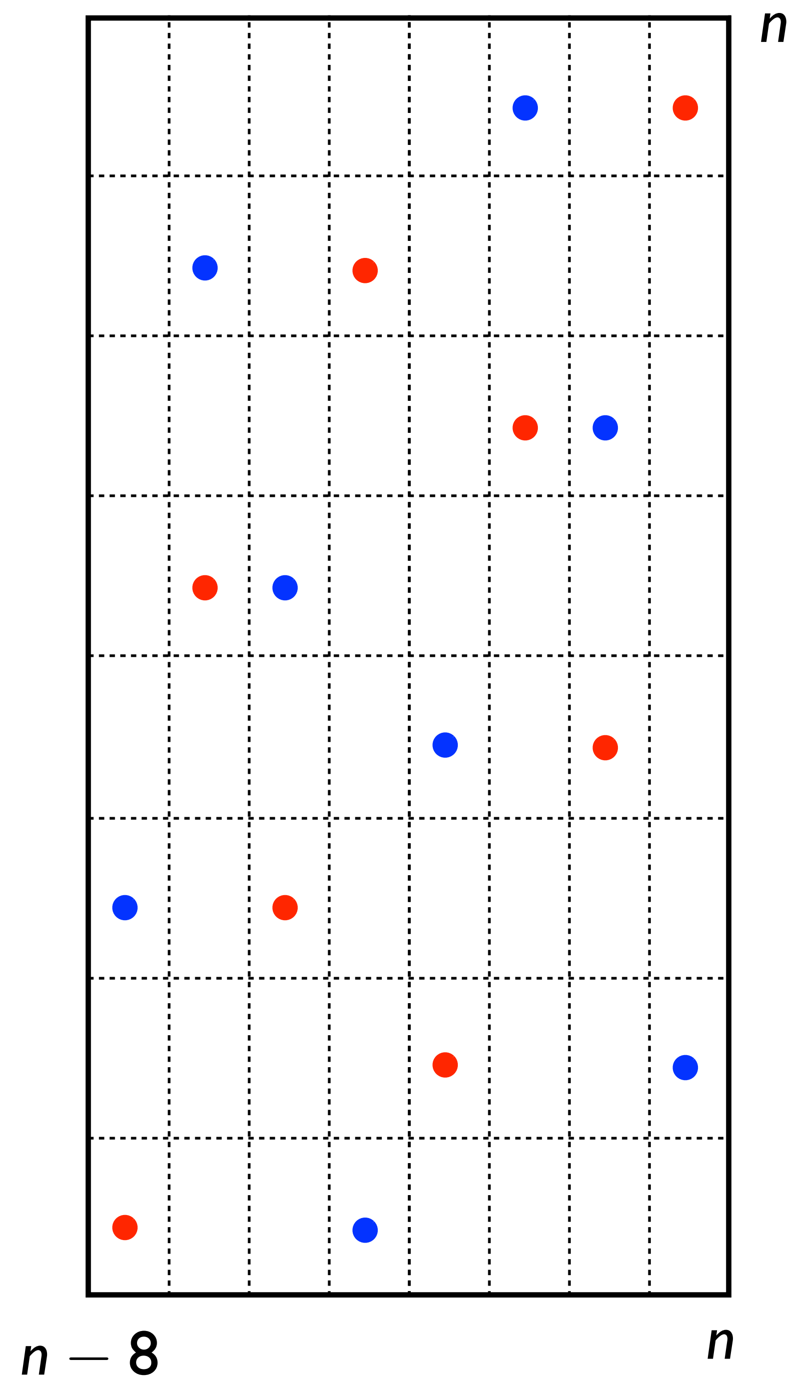


$$Z_1 = \begin{cases} 1 & \text{(red)} \\ 0 & \text{(blue)} \end{cases} \quad \Delta_1 \geq \frac{n}{16}$$

Binary Nets with Large CVD

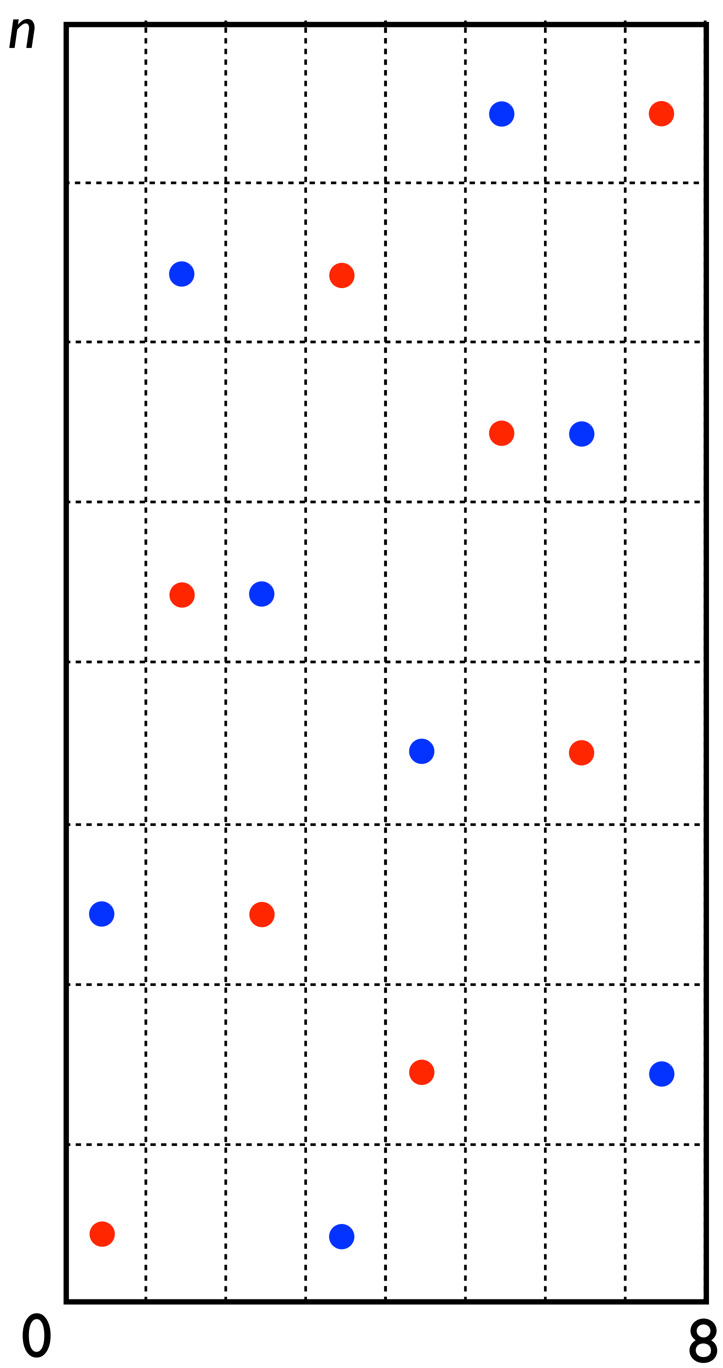


. . .



$Z_1 = \begin{cases} 1 & \text{(red)} \\ 0 & \text{(blue)} \end{cases}$ $\quad \Delta_1 \geq \frac{n}{16}$

Binary Nets with Large CVD



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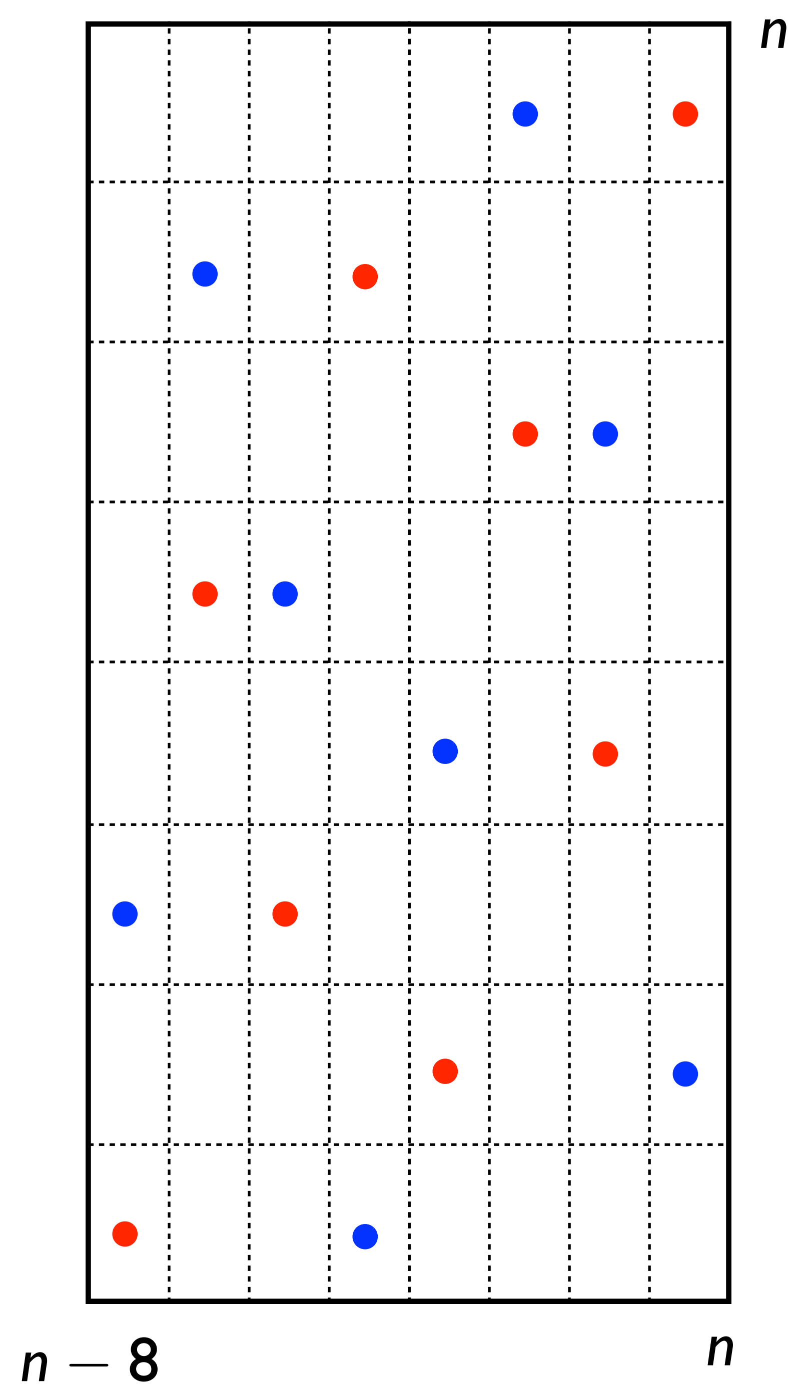
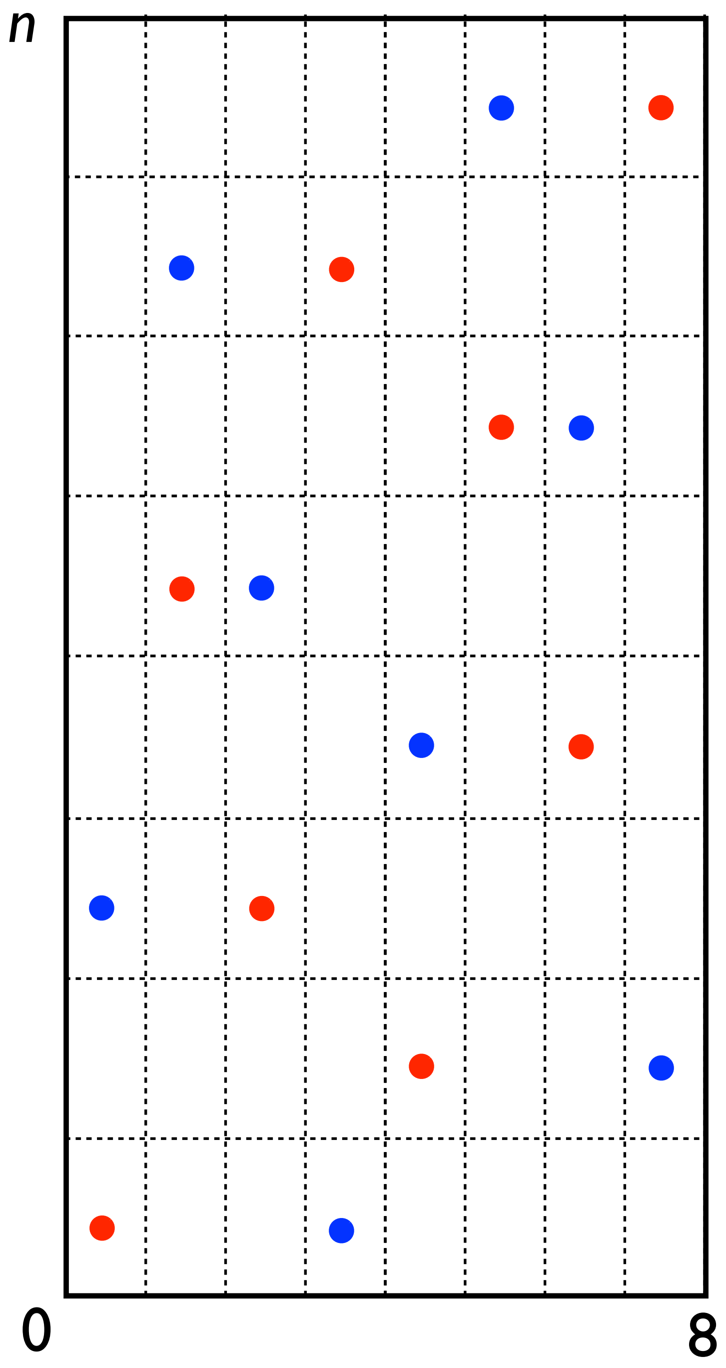


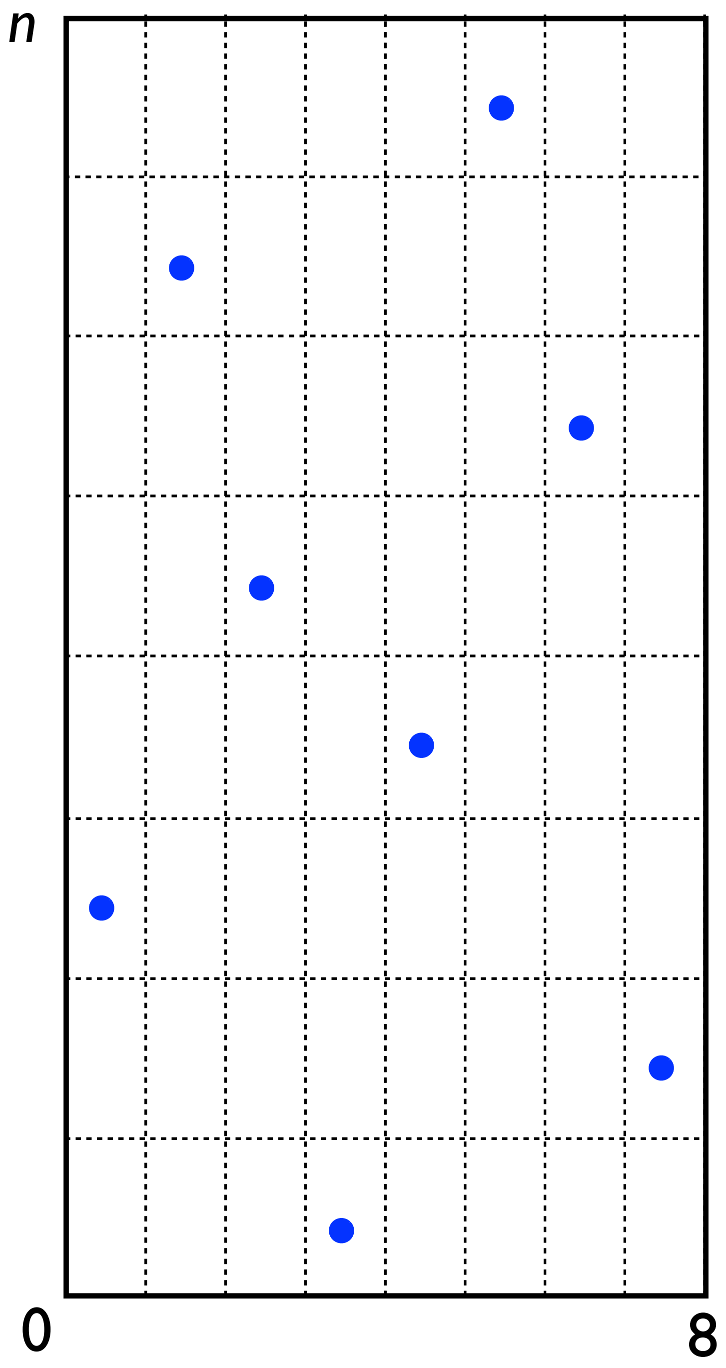
$$Z_1 = \begin{cases} 1 & {\color{red}\bullet} \\ 0 & {\color{blue}\bullet} \end{cases} \quad \Delta_1 \geq \frac{n}{16}$$

$$Z_{\frac{n}{8}} = \begin{cases} 1 & {\color{red}\bullet} \\ 0 & {\color{blue}\bullet} \end{cases} \quad \Delta_{\frac{n}{8}} \geq \frac{n}{16}$$

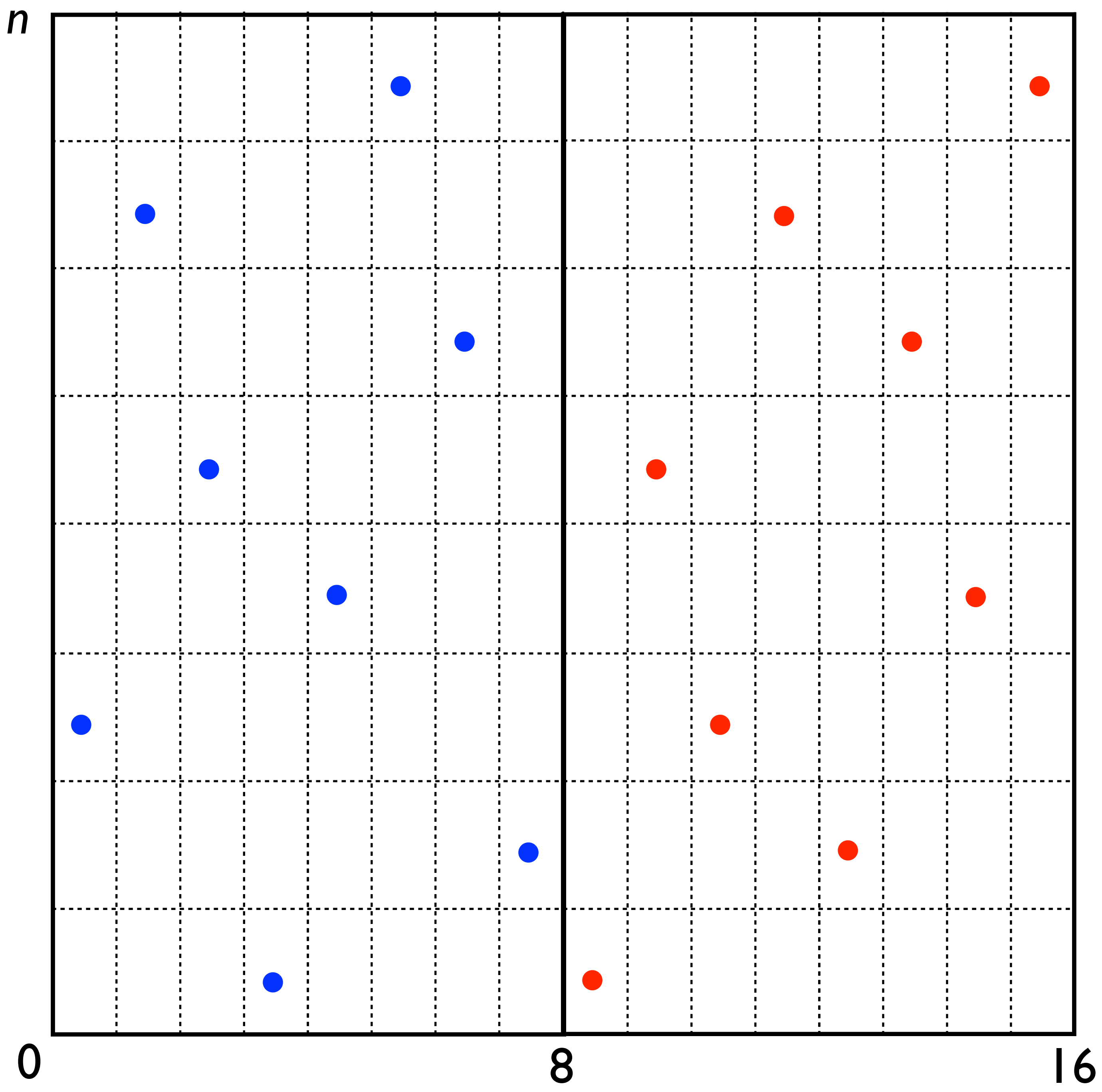
Binary Nets with Large CVD



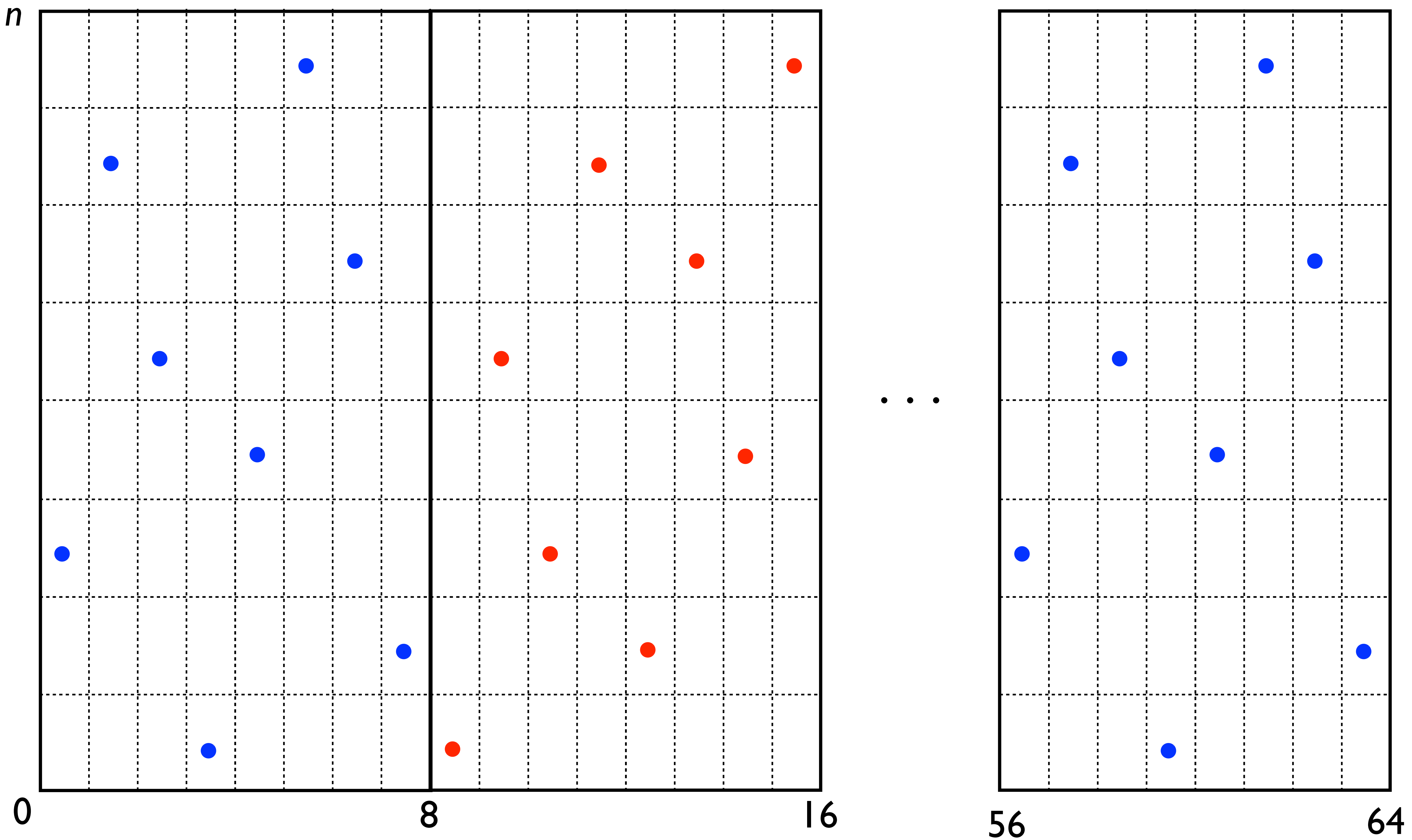
Binary Nets with Large CVD



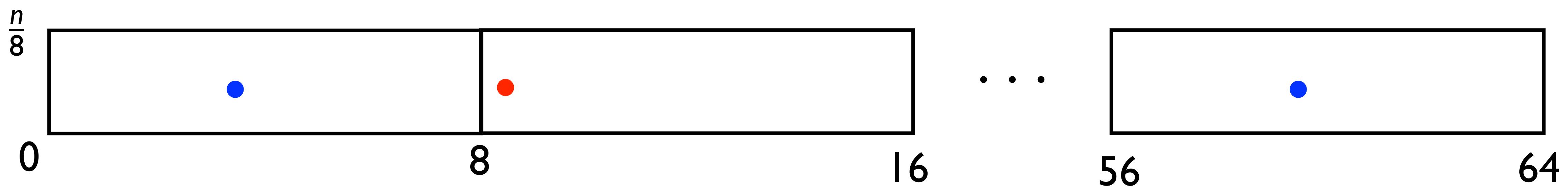
Binary Nets with Large CVD



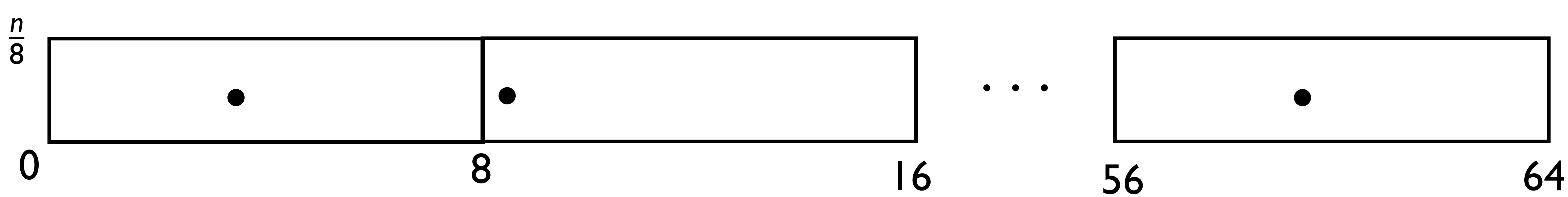
Binary Nets with Large CVD



Binary Nets with Large CVD

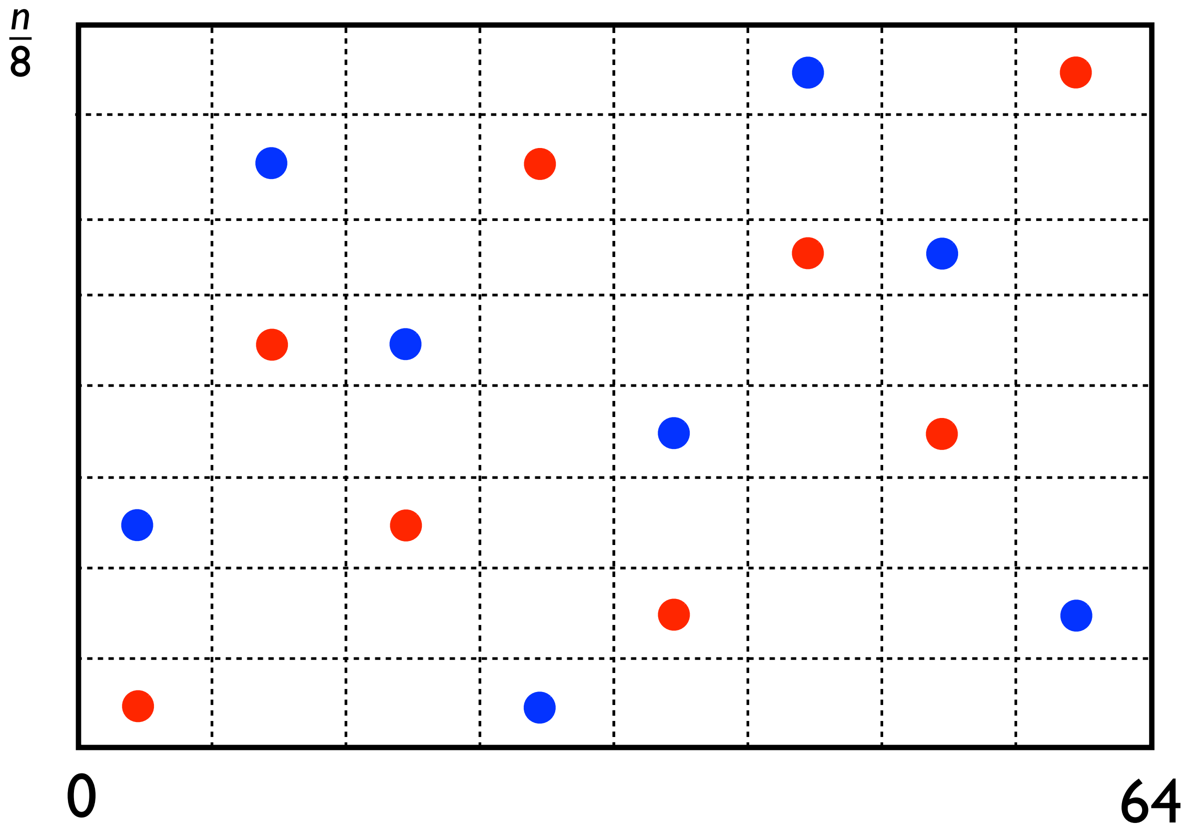


Binary Nets with Large CVD



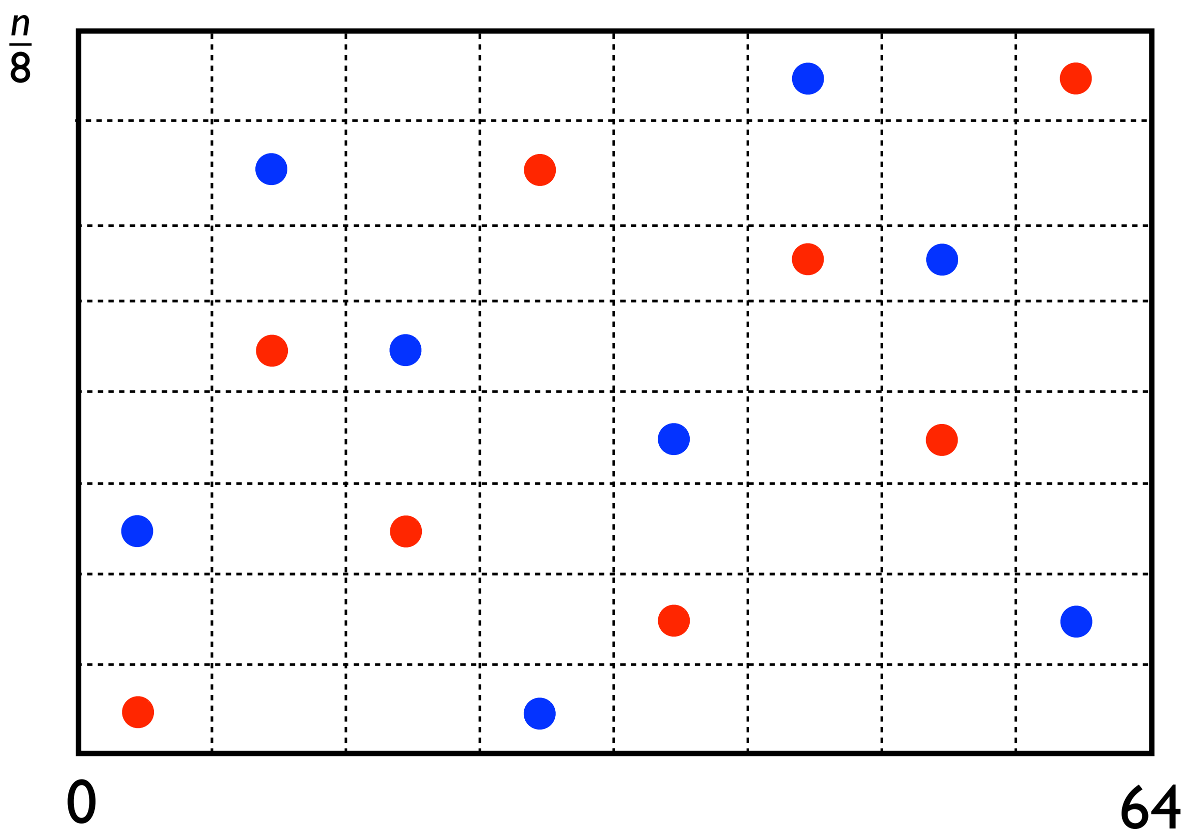
Layer 3 to Layer 7

Binary Nets with Large CVD

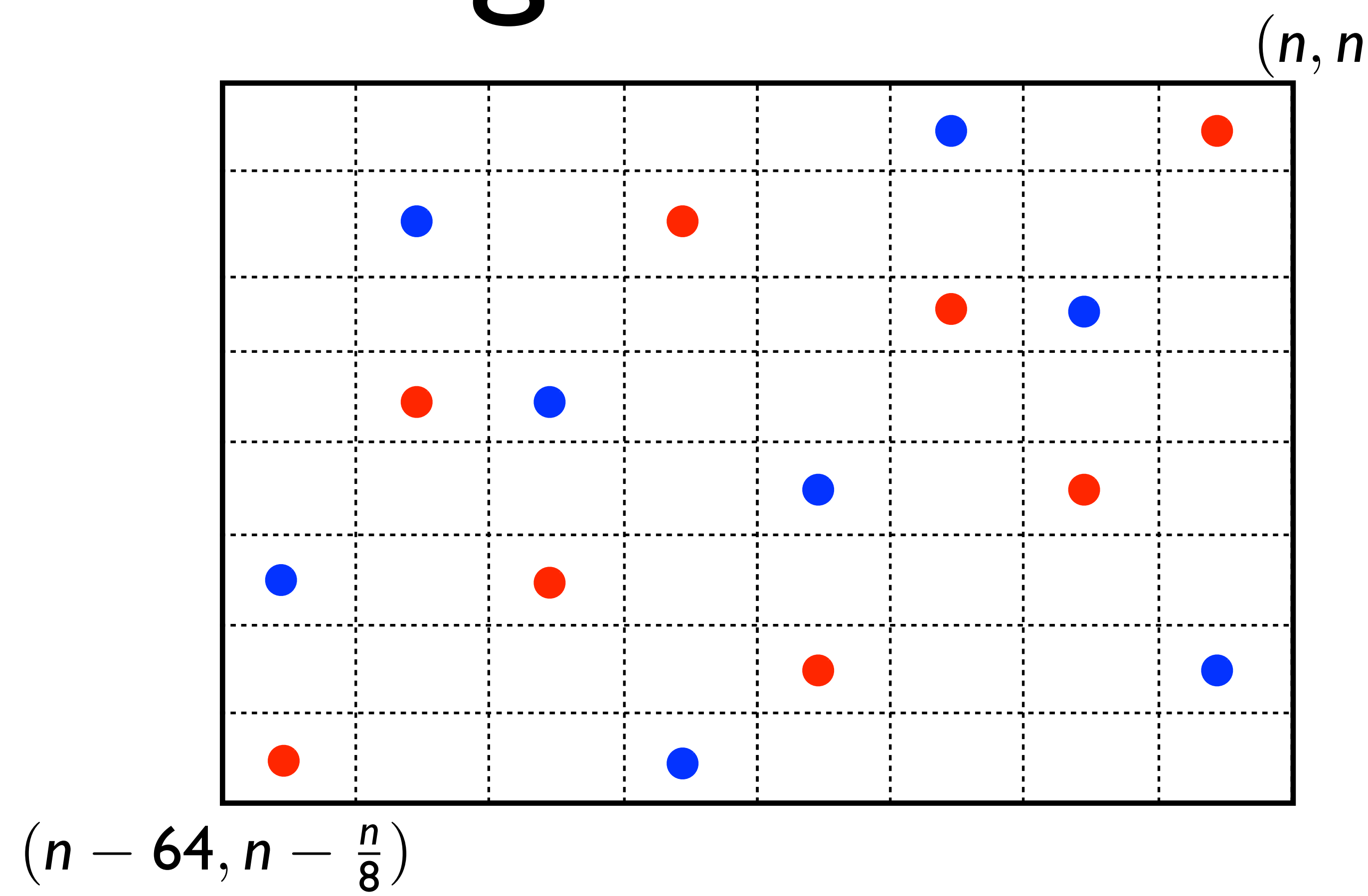


Binary Nets with Large CVD

$$Z_{\frac{n}{8}+1} = \begin{cases} 1 & \bullet \\ 0 & \bullet \end{cases} \quad \Delta_{\frac{n}{8}+1} \geq \frac{n}{16}$$

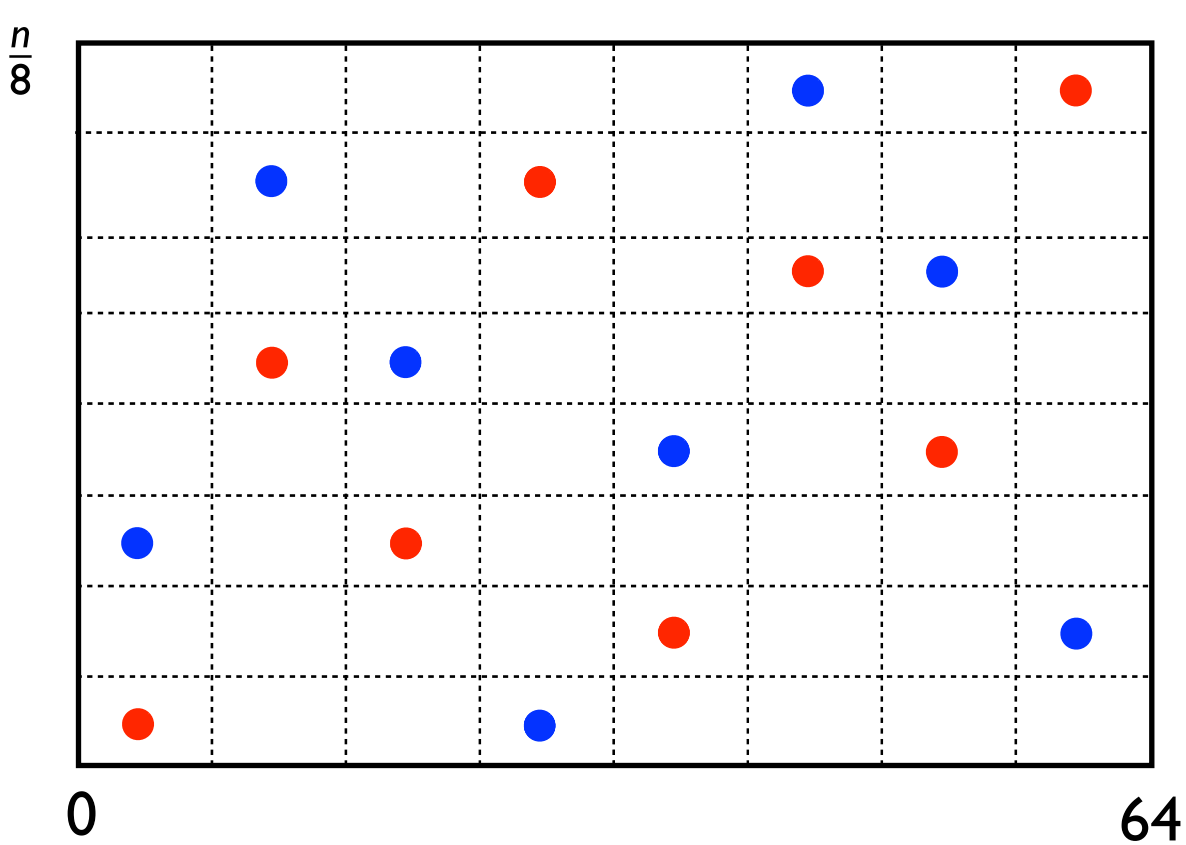


Binary Nets with Large CVD



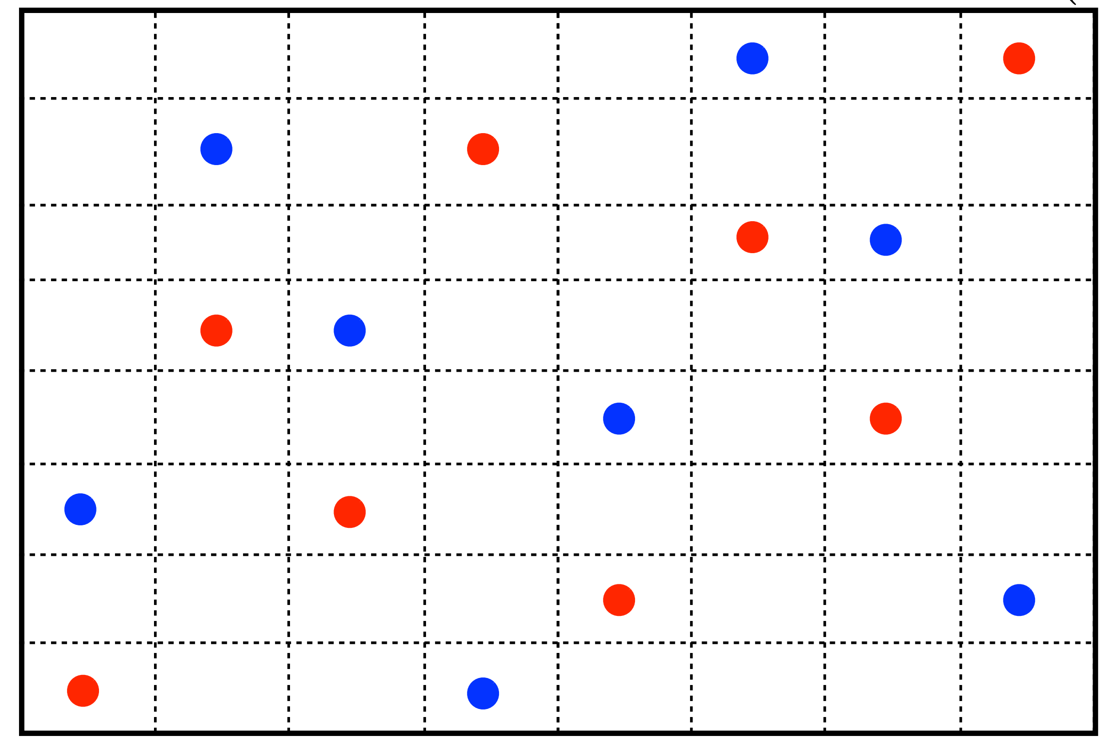
$$Z_{\frac{n}{8}+1} = \begin{cases} 1 & \bullet \\ 0 & \bullet \end{cases} \quad \Delta_{\frac{n}{8}+1} \geq \frac{n}{16}$$

. . .



Binary Nets with Large CVD

(n, n)



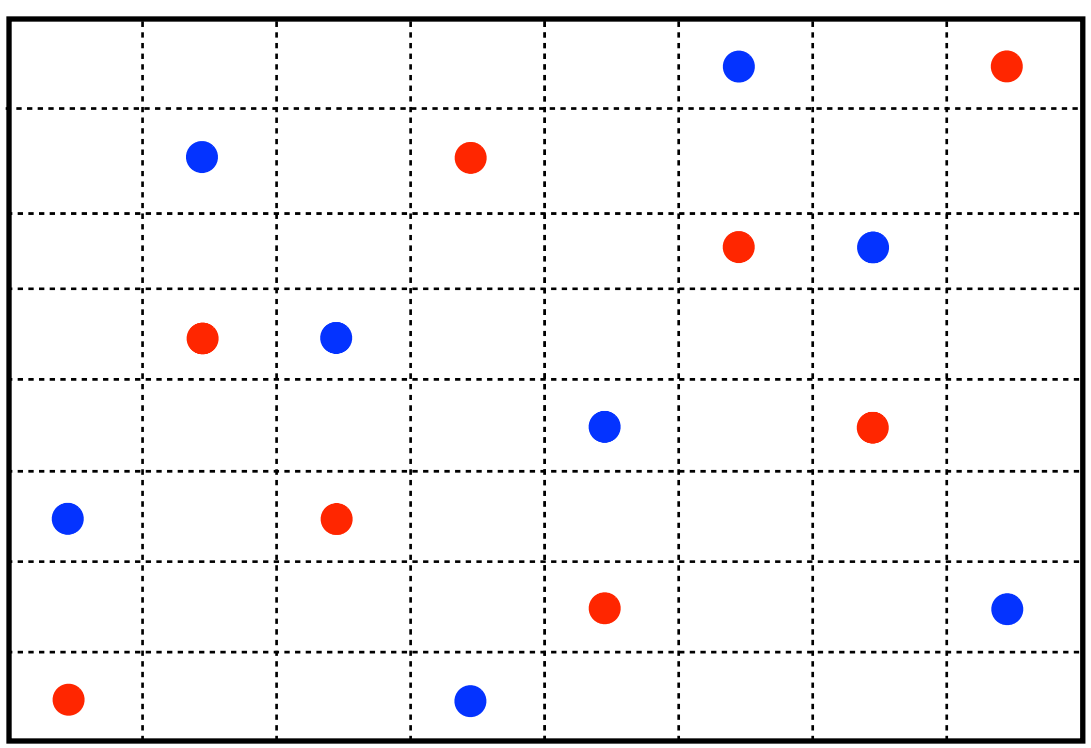
$(n-64, n-\frac{n}{8})$

$\cdots$

$$Z_{\frac{n}{8}+\frac{n}{8}} = \begin{cases} 1 & \text{(red)} \\ 0 & \text{(blue)} \end{cases} \qquad \Delta_{\frac{n}{8}+\frac{n}{8}} \geq \frac{n}{16}$$

$$Z_{\frac{n}{8}+1} = \begin{cases} 1 & \text{(red)} \\ 0 & \text{(blue)} \end{cases} \qquad \Delta_{\frac{n}{8}+1} \geq \frac{n}{16}$$

$\frac{n}{8}$



0

64

Binary Nets with Large CVD

$$Z_1 = \begin{cases} 1 \\ 0 \end{cases} \quad \Delta_1 \geq \frac{n}{16} \qquad \cdots \qquad Z_{\frac{n}{8}\cdot\frac{\log n}{4}} = \begin{cases} 1 \\ 0 \end{cases} \quad \Delta_{\frac{n}{8}\cdot\frac{\log n}{4}} \geq \frac{n}{16}$$

Binary Nets with Large CVD

$$Z_1 = \begin{cases} 1 & \textcolor{red}{\bullet} \\ 0 & \textcolor{blue}{\bullet} \end{cases} \quad \Delta_1 \geq \frac{n}{16} \qquad \cdots \qquad Z_{\frac{n}{8} \cdot \frac{\log n}{4}} = \begin{cases} 1 & \textcolor{red}{\bullet} \\ 0 & \textcolor{blue}{\bullet} \end{cases} \quad \Delta_{\frac{n}{8} \cdot \frac{\log n}{4}} \geq \frac{n}{16}$$

- Let $\mathbf{Z} = (Z_1, \ldots, Z_{\frac{n \log n}{32}})$.

Binary Nets with Large CVD

$Z_1 = \begin{cases} 1 & \text{(red)} \\ 0 & \text{(blue)} \end{cases} \quad \Delta_1 \geq \frac{n}{16} \qquad \cdots \qquad Z_{\frac{n}{8}\cdot\frac{\log n}{4}} = \begin{cases} 1 & \text{(red)} \\ 0 & \text{(blue)} \end{cases} \quad \Delta_{\frac{n}{8}\cdot\frac{\log n}{4}} \geq \frac{n}{16}$

- Let $\mathbf{Z} = (Z_1, \ldots, Z_{\frac{n \log n}{32}})$.

An assignment of $\mathbf{Z}$ $\longleftrightarrow$ A binary net

Binary Nets with Large CVD

$Z_1 = \begin{cases} 1 & \textcolor{red}{\bullet} \\ 0 & \textcolor{blue}{\bullet} \end{cases} \quad \Delta_1 \geq \frac{n}{16} \qquad \cdots \qquad Z_{\frac{n}{8}\cdot\frac{\log n}{4}} = \begin{cases} 1 & \textcolor{red}{\bullet} \\ 0 & \textcolor{blue}{\bullet} \end{cases} \quad \Delta_{\frac{n}{8}\cdot\frac{\log n}{4}} \geq \frac{n}{16}$

- Let $\mathbf{Z} = (Z_1, \ldots, Z_{\frac{n \log n}{32}})$.

An assignment of $\mathbf{Z}$	$\longleftrightarrow$	A binary net
Vector space $\{0, 1\}^{\frac{n \log n}{32}}$	$\longleftrightarrow$	$\|\mathcal{P}_1\| = 2^{\frac{n \log n}{32}}$

Binary Nets with Large CVD

$$Z_1 = \begin{cases} 1 & \textcolor{red}{\bullet} \\ 0 & \textcolor{blue}{\bullet} \end{cases} \quad \Delta_1 \geq \frac{n}{16} \qquad \cdots \qquad Z_{\frac{n}{8}\cdot\frac{\log n}{4}} = \begin{cases} 1 & \textcolor{red}{\bullet} \\ 0 & \textcolor{blue}{\bullet} \end{cases} \quad \Delta_{\frac{n}{8}\cdot\frac{\log n}{4}} \geq \frac{n}{16}$$

- Let $\mathbf{Z} = (Z_1, \ldots, Z_{\frac{n \log n}{32}})$.

An assignment of $\mathbf{Z}$	$\longleftrightarrow$	A binary net
Vector space $\{0, 1\}^{\frac{n \log n}{32}}$	$\longleftrightarrow$	$\lvert\mathcal{P}_1\rvert = 2^{\frac{n \log n}{32}}$
$H(\mathbf{Z}(P_1), \mathbf{Z}(P_2)) \geq cn \log n$	$\longleftrightarrow$	$\Delta(P_1, P_2) \geq \frac{cn^2}{16} \log n$

Binary Nets with Large CVD

$Z_1 = \begin{cases} 1 & \textcolor{red}{\bullet} \\ 0 & \textcolor{blue}{\bullet} \end{cases} \quad \Delta_1 \geq \frac{n}{16} \qquad \cdots \qquad Z_{\frac{n}{8}\cdot\frac{\log n}{4}} = \begin{cases} 1 & \textcolor{red}{\bullet} \\ 0 & \textcolor{blue}{\bullet} \end{cases} \quad \Delta_{\frac{n}{8}\cdot\frac{\log n}{4}} \geq \frac{n}{16}$

- Let $\mathbf{Z} = (Z_1, \ldots, Z_{\frac{n \log n}{32}})$.

An assignment of $\mathbf{Z}$	$\longleftrightarrow$	A binary net
Vector space $\{0,1\}^{\frac{n \log n}{32}}$	$\longleftrightarrow$	$\lvert\mathcal{P}_1\rvert = 2^{\frac{n \log n}{32}}$
$H(\mathbf{Z}(P_1), \mathbf{Z}(P_2)) \geq cn \log n$	$\longleftrightarrow$	$\Delta(P_1, P_2) \geq \frac{cn^2}{16} \log n$

- Large subspace of $\{0,1\}^{\frac{n \log n}{32}}$ with large mutual hamming distance?

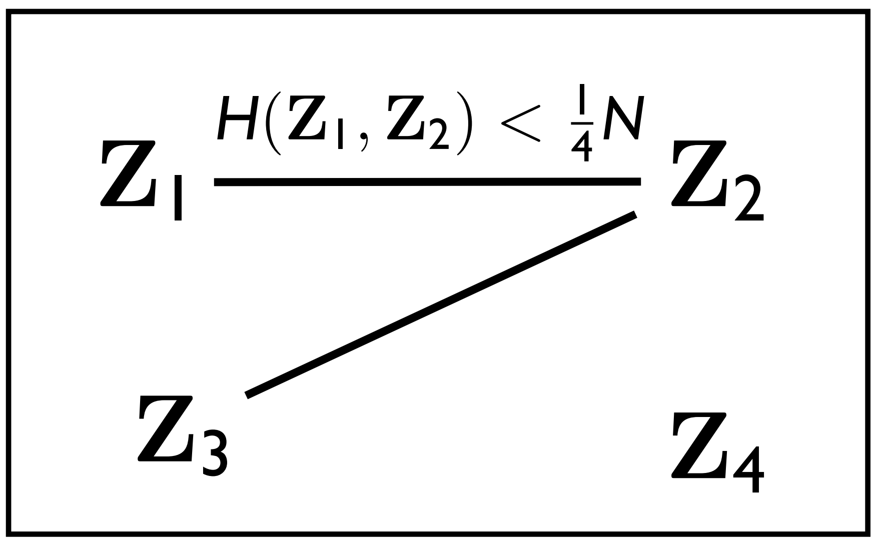
Binary Nets with Large CVD

- $N = \frac{n \log n}{32}$. $\exists \mathcal{Z} \subset \{0,1\}^N$, s.t. $|\mathcal{Z}| = 2^{\Omega(N)}$ and $\forall \mathbf{Z}_1, \mathbf{Z}_2 \in \mathcal{Z}$, $H(\mathbf{Z}_1, \mathbf{Z}_2) = \Omega(N)$.

Binary Nets with Large CVD

- $N = \frac{n \log n}{32}$. $\exists \mathcal{Z} \subset \{0,1\}^N$, s.t. $|\mathcal{Z}| = 2^{\Omega(N)}$ and $\forall \mathbf{Z}_1, \mathbf{Z}_2 \in \mathcal{Z}$, $H(\mathbf{Z}_1, \mathbf{Z}_2) = \Omega(N)$.

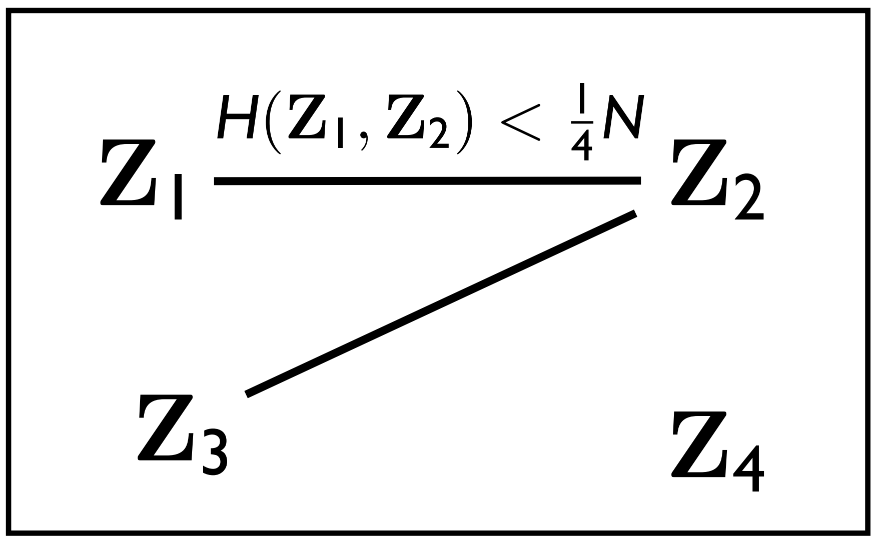
Graph



Binary Nets with Large CVD

- $N = \frac{n \log n}{32}$. $\exists \mathcal{Z} \subset \{0, 1\}^N$, s.t. $|\mathcal{Z}| = 2^{\Omega(N)}$ and $\forall \mathbf{Z}_1, \mathbf{Z}_2 \in \mathcal{Z}$, $H(\mathbf{Z}_1, \mathbf{Z}_2) = \Omega(N)$.

Graph

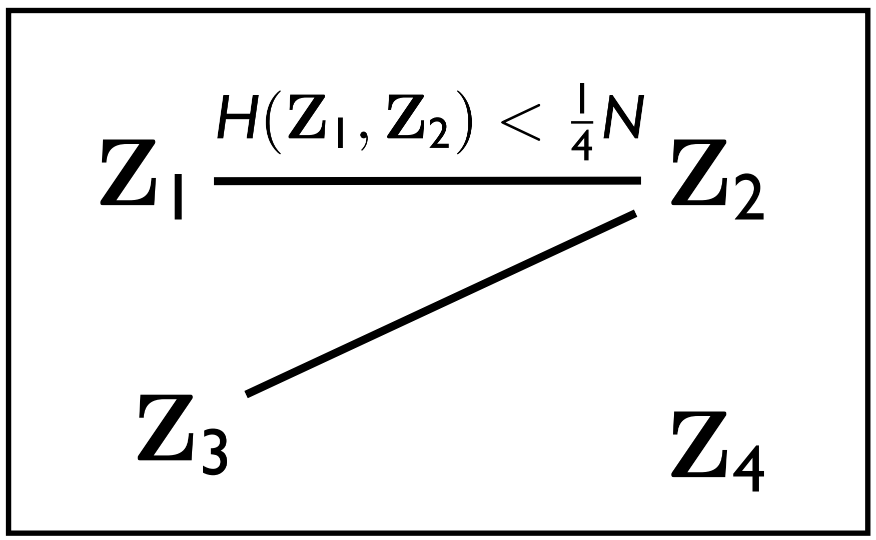


2^N vertices

Binary Nets with Large CVD

- $N = \frac{n \log n}{32}$. $\exists \mathcal{Z} \subset \{0,1\}^N$, s.t. $|\mathcal{Z}| = 2^{\Omega(N)}$ and $\forall \mathbf{Z}_1, \mathbf{Z}_2 \in \mathcal{Z}$, $H(\mathbf{Z}_1, \mathbf{Z}_2) = \Omega(N)$.

Graph



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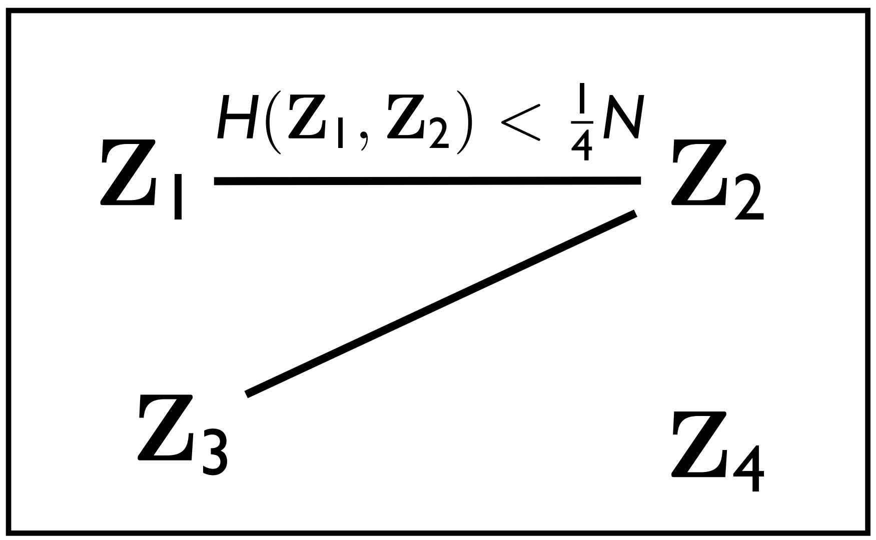

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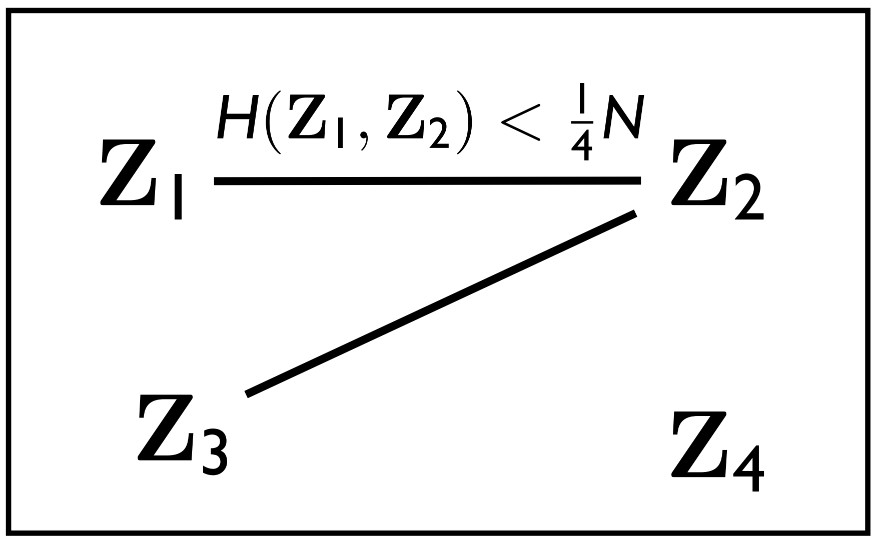

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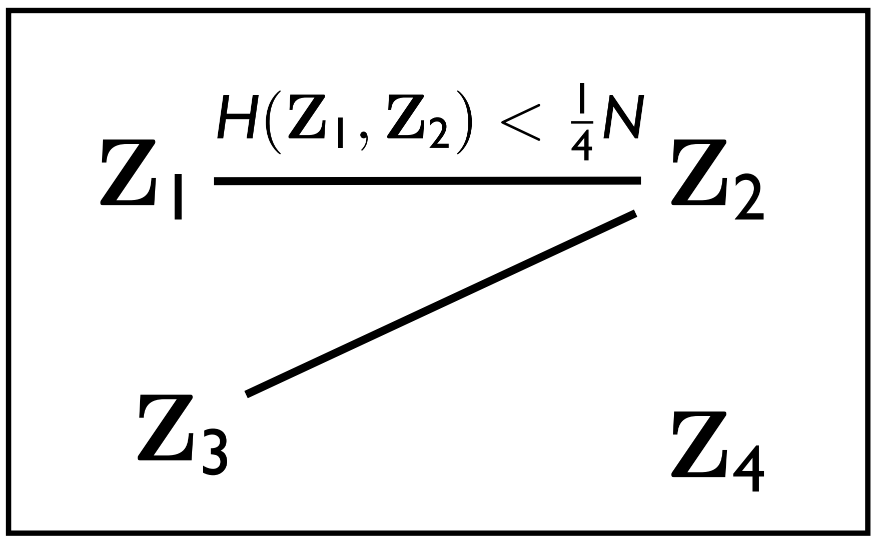

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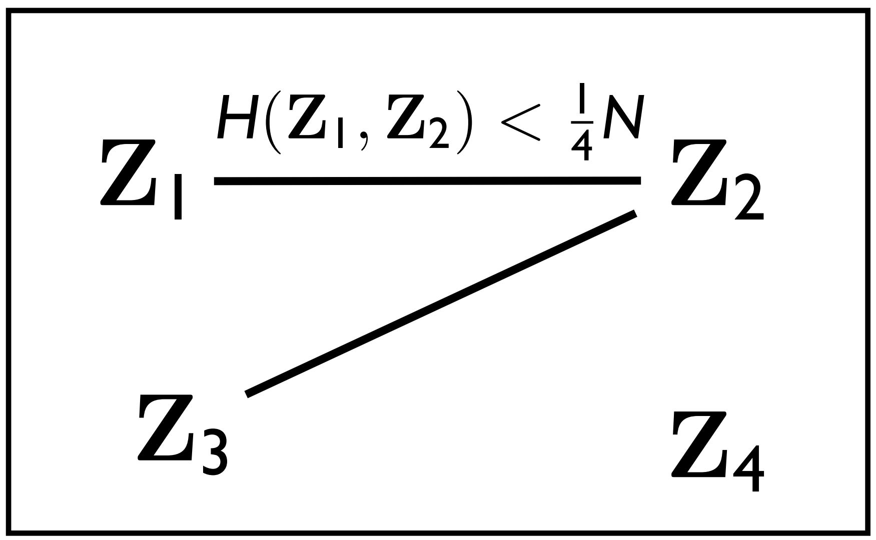

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Thank you!