Bayesian Nash Equilibrium in First-Price Auction with Discrete Value Distributions

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ABSTRACT

First price auctions are widely used in government contracts and ads auctions. In this paper, we consider the Bayesian Nash Equilibrium (BNE) in first price auctions with discrete value distributions. We study the characterization of the BNE in the first price auction and provide an algorithm to compute the BNE at the same time. Moreover, we prove the existence and the uniqueness of the BNE. Some of the previous results in the case of continuous value distributions do not apply to the case of discrete value distributions. In the meanwhile, the uniqueness result in discrete case cannot be implied by the uniqueness property in the continuous case. Unlike in the continuous case, we do not need to solve ordinary differential equations and thus do not suffer from the solution errors therein. Compared to the method of using continuous distributions to approximate discrete ones, our experiments show that our algorithm is both faster and more accurate.

The results in this paper are derived in the asymmetric independent private values model, which assumes that the buyers' value distributions are common knowledge.

KEYWORDS

first price auction, Nash equilibrium, discrete distribution

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1 INTRODUCTION

Recently, the display advertising industry has switched from second-price to first-price auctions,[1] and one important reason is that some advertisers no longer trust the exchange to honestly follow the second-price auction rules [24]. From a buyer's point of view, he does not trust the auctioneer since the auctioneer could also benefit from manipulating the auction rules after observing the sealed bids [1]. Akbarpour and Li [1] show that the first-price auction is the unique *credible* and *static* optimal auction, which may be one potential backing theory for the trend of adopting the first-price auction in the ad exchange industry. Akbarpour and Li [1] also prove that no mechanism is static, credible, and strategy-proof (incentive compatible) at the same time. In particular, being credible means that it is incentive compatible for the auctioneer to follow the rules and being static roughly means that the auction is sealed-bid. Therefore, the first-price auction naturally becomes the only choice for the ad exchange industry, in which credibility becomes a major concern and sealed-bid is also critical to keep the auction process time-efficient for production needs.

In contrast to the crucial needs from practice, the understanding of the first-price auction from auction theory remains shallow. The essential obstacle is the complex equilibrium structure in first-price auctions. Following the first step by Vickrey [25] for the symmetric setting, it has been a tough and long journey towards the existence, the uniqueness and the computation of the Bayesian Nash equilibrium of first-price auctions in general settings. Plum [21] covers the power distribution $G_1(x) = x^\mu$ and $G_2(x) = (\frac{x}{\beta})^\mu$ with the same support. Kaplan and Zamir [12] solve the problem for uniform distributions with different support.

In this paper, with the application in ad auctions as one of the important motivations, we focus on the computation of BNEs in first-price auctions where the buyers' values are independently drawn from discrete prior distributions. We study the discrete value setting for several reasons. First, it is questionable that buyers are always assumed to have continuous distribution. Where does this continuous distribution come from? Furthermore, from a practical point of view, how should we generate such a distribution since we could only get a finite number of observations about a buyer's history actions? Second, discrete distribution is a basic setting in which the BNE shows an elaborated structure. The results in the discrete setting can provide us with more structural insights that can not be obtained from the continuous case. Third, discrete distributions can be used to approximate the general continuous distribution, our algorithm for computing the BNE under discrete distribution can also be applied under continuous distribution.

1.1 Our Contributions

We give an efficient algorithm to find the BNE of the first price auction. Compared to the algorithms developed for continuous distribution, our algorithm does not involve ordinary differential equations, which makes our algorithm robust and much faster. For any possible bid, by scrutinizing the buyers who might report it, we give a clear characterization of the BNE in the discrete setting. Previous methods make use of Nash's Theorem to prove the existence of the equilibrium in the continuous case, while we provide a constructive proof in the discrete case.

*Part of this work was done while this author was a Ph.D. candidate at Tsinghua University.
[1]The winners now *pay the bid price* in the *Exchange Bidding* auctions by Google Ad Manager (Google's display ad-serving system for publishers). In Exchange Bidding auctions, the exchanges bid to compete for ad requests from publishers https://support.google.com/admanager/answer/7128958.

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We show that the equilibrium is unique in the discrete case (Theorem 5.4). The uniqueness result by Lebrun [15] relies on a technical assumption about buyers' value distributions. In contrast, we do not need any assumption. Furthermore, in the continuous case, we need to be very careful when a buyer's value is near the smallest value. In the discrete case, each buyer's strategy around the smallest value has relatively nice properties.

1.2 Related Works

Besides the closed-form solution of the equilibrium, there is also a line of papers that focus on other aspects of the problem [20, 22, 23]. Lebrun [13] first proves the existence of a Bayesian Nash equilibrium under a distribution assumption. Later Athey [3], Maskin and Riley [18] prove the existence of equilibrium without distribution assumption. Maskin and Riley [18] show the existence for discrete distributions by applying Nash's Theorem. To show the existence in the continuous case, they constructed a series of discrete distributions that approaches the actual continuous case. In this paper, we prove the existence result by construction.

After proving the existence of the BNE, researchers began to consider the its uniqueness. For symmetric distributions, Chawla and Hartline [5] prove the uniqueness by ruling out asymmetric equilibria. For asymmetric distributions, Maskin and Riley [19] show that the equilibrium is unique for asymmetric distributions with the assumption that buyers' distributions share the same upper endpoint. Lebrun [14, 15] prove the uniqueness for more general settings but still with the assumption that the cumulative value distribution functions are strictly log-concave at certain points. Escamocher et al. [6] investigate the existence and the computation of BNEs in the discrete case, under the assumption that buyers can only place discrete bids. They consider both the randomized tie-breaking and the Vickery tie-breaking and give different results.

However, both the continuous and the discrete case without assumptions are still left open. In this paper, we solve the discrete value distribution case.

In the numerical analysis literature, Marshall et al. [17] give the first numerical analysis for two special distributions. Their backward-shooting method then become the standard method for computing the equilibrium strategies of asymmetric first-price auctions [4, 7, 16]. The backward-shooting method first computes the smallest winning bid, then repeatedly guess the largest winning bid and then solving ordinary differential equations all the way down in the bid space to see if the smallest winning bid given by the solution to the differential equations matches the actual one. One common issue of this method is the computation error in solving ordinary differential equations. Bajari [4] uses a polynomial to approximate the inverse bidding strategy. To compute a solution with high precision, Gayle and Richard [10] use Taylor-series expansions. Our method belongs to the backward-shooting category. We do not need to solve ordinary differential equations, but the algorithm still needs to repeatedly guess the largest winning bid. Fibich and Gavish [9] propose a forward-shooting method and numerically solve the case with power-law distributions. However this forward-shooting method does not work in the discrete case. Armantier et al. [2] use a sequence of constrained strategic equilibrium to approximate the exact equilibrium. This method can be applied to a broad class of Bayesian games. Hubbard and Paarsch [11] give a comprehensive survey in numerical analysis literature.

2 PRELIMINARIES

2.1 Model

Suppose the seller has one item for sale and there are n potential buyers $N = \{1, \ldots, n\}$. The item is sold through a sealed-bid first-price auction. Each buyer has a private value for the item, which is drawn according to a publicly known value distribution. In our setting, we consider the case where the each buyer's value distribution is discrete. Also, we assume that for buyer i, the value support is a finite set $\{v_i^1, v_i^2, \ldots, v_i^{d_i}\}$ with cumulative distribution function G_i, i.e., $G_i(v) = \text{Prob}\{v_i \leq v\}$. Without loss of generality, we assume $0 \leq v_i^1 < v_i^2 < \ldots < v_i^{d_i}$.

Every buyer places a nonnegative bid b_i simultaneously. Let $F_i(b_i)$ denote the cumulative distribution function of buyer i's bids. For simplicity, we assume that buyers have quasi-linear utilities and no buyer overbids, i.e., no buyer will place a bid that is higher than his value. The buyer with the highest bid wins the item and pays what he bids. Each buyer's strategy is a mapping from his private value to his bid. The strategies form a Bayesian Nash Equilibrium (BNE) if no bidder has an incentive to change his strategy unilaterally in the Bayesian setting.

In the continuous value setting, each buyer's strategy maps a value to a bid. For example, suppose there are two i.i.d. buyers with value uniformly distributed between $[0, 1]$. In the BNE, each buyer bids half of his private value. But in the discrete setting, each buyer's strategy is randomized, and maps a value to a set of possible bids, with a certain probability distribution.

Example 2.1. There are two i.i.d. buyers. Each buyer has value 1 and 2 with probability 0.5. In the equilibrium, when a buyer's value is 2, it is possible for him to place any bid in $[1, 1.5]$, and the bid density function is $\frac{1}{(2-x)^2}, \forall x \in [1, 1.5]$. When a buyer's value is 1, the buyer bids 1 with probability 1.

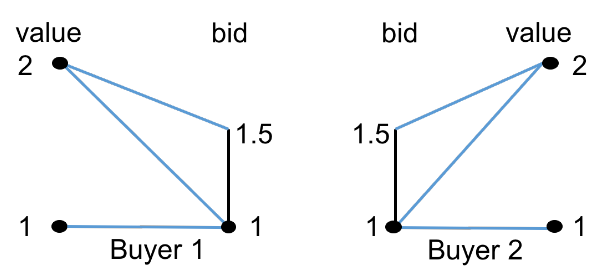


Figure 1: The equilibrium strategy of two i.i.d. buyers with uniform $\{1, 2\}$ value distribution. Although the value distribution is discrete, the bids are continuous.

Our objective is to find the bidding strategies that constitute a Bayesian Nash equilibrium. Before we start, we need to make an assumption of the tie-breaking rule to guarantee the existence of an equilibrium (see Example 2 in Maskin and Riley [18], where an equilibrium does not exist). We consider an alternative tie-breaking rule introduced by Maskin and Riley [18]. When there are multiple highest bids, we will allocate the item to the buyer with the highest value.

Assumption 1 (Maskin and Riley [18]). *Ties are broken by running a Vickrey auction among the highest buyers.*[2]

In the continuous value setting, this assumption is unnecessary, but in the discrete value setting, we need this assumption to deal with best response issue. Without this assumption, we will still get an approximate BNE using our algorithm. We will discuss more about this assumption (see Example 2.6).

2.2 Basic Structure of the BNE

To assist later arguments, we restate several properties of the Bayesian Nash equilibrium, mainly summarized by Maskin and Riley [19].[3] Giving all buyers' strategies, if any buyer can win with a certain probability by bidding b then we call b a *winning bid*. Without loss of generality, we assume the set of winning bids is closed and there exists a smallest winning bid $\underline{b}$. As long as a buyer's bid is higher than or equal to $\underline{b}$, he can win with a certain probability. If buyers' strategies form a BNE, the smallest winning bid $\underline{b}$ will be determined uniquely by the buyers' value distributions:

Lemma 2.2 (Maskin and Riley [19]). *Assume buyer i^* has the largest smallest value, i.e., $v_{i^*}^1 = \max_j v_j^1$. Then the smallest winning bid is $\underline{b} = \arg\max_b (v_{i^*}^1 - b) \prod_{i \neq i^*} G_i(b)$.*

It is possible that a buyer with a certain value v_i^j may place multiple bids. Let $S_i(v_i^j)$ be the set of possible bids for buyer i when he has value v_i^j. For ease of presentation, we assume that $S_i(v_i^j)$ is a closed set, otherwise, we can take the closure of the support as $S_i(v_i^j)$. Denote buyer i's all possible bids by S_i, i.e., $S_i = \bigcup_j S_i(v_i^j)$.

Maskin and Riley [19] show that for any winning bid b, there are at least two buyers i and j, such that $b \in S_i$ and $b \in S_j$. The intuition is that any buyer who places a winning bid b needs a competitor, otherwise, the buyer can place $b - \epsilon$ to increase his utility.

Lemma 2.3 (Maskin and Riley [19]). *In the BNE of the first-price auction, for buyer i and any $b_i > \underline{b}$, if $b_i \in S_i$, then there must exist another buyer, who bids in $(b_i - \epsilon, b_i)$ with positive probability for any ϵ.*

Maskin and Riley [19] show that, in first price auctions, a buyer would not give a particular bid with positive probability when this bid is larger than or equal to $\underline{b}$.

Lemma 2.4 (Maskin and Riley [19]). *For any buyer i, there is no mass point above $\underline{b}$ in buyer i's bid distribution.*

The following lemma shows that when a buyer's value is larger than or equal to $\underline{b}$, his bidding strategy is monotone in his value.

Lemma 2.5 (Maskin and Riley [19]). *For each buyer, his bidding strategy is monotone in value, i.e., $\max S_i(v_i^j) \leq \min S_i(v_i^{j+1})$ for $v_i^j \geq \underline{b}$.*

[2]This can be implemented by letting the highest buyers submit new bids.
[3]Although they assume twice continuously differentiable value distributions and the buyers have the same upper limit of values, the lemmas still hold for the present setting.

Here we provide an example of what the BNE looks like in the discrete case. The computation of such a BNE will be clear after the analysis of our algorithm.

Example 2.6. Suppose there are 4 buyers with the following discrete value distributions:

x	$G_1(x) =$
20	1
10	$11\sqrt{7}/24\sqrt{3}$
2	$\sqrt{77}/12\sqrt{2}$

x	$G_2(x) =$
14	1
13	$4/\sqrt{21}$
1	$2\sqrt{22}/7\sqrt{7}$

x	$G_3(x) =$
20	1
9	$11/12$

x	$G_4(x) =$
12	1
1	$3\sqrt{3}/2\sqrt{7}$

In the BNE, the buyers bid according to the following bid distributions:

$x \in$	$F_1(x) =$
$(8, 9]$	$\frac{11}{20-x}$
$(6, 8]$	$\frac{11}{12}\sqrt{\frac{2(20-x)}{(14-x)(12-x)}}$
$[2, 6]$	$\frac{77}{48}\sqrt{\frac{10-x}{(9-x)(13-x)}}$

$x \in$	$F_2(x) =$
$(6, 8]$	$\sqrt{\frac{8(14-x)}{(20-x)(12-x)}}$
$[2, 6]$	$\frac{8}{7}\sqrt{\frac{13-x}{(10-x)(9-x)}}$
$\{1\}$	$\frac{2\sqrt{22}}{7\sqrt{7}}$

$x \in$	$F_3(x) =$
$(8, 9]$	$\frac{11}{20-x}$
$[2, 6]$	$\frac{11}{6}\sqrt{\frac{7(9-x)}{3(10-x)(13-x)}}$

$x \in$	$F_4(x) =$
$[6, 8]$	$\sqrt{\frac{18(12-x)}{(20-x)(14-x)}}$
$\{1\}$	$\frac{3\sqrt{3}}{2\sqrt{7}}$

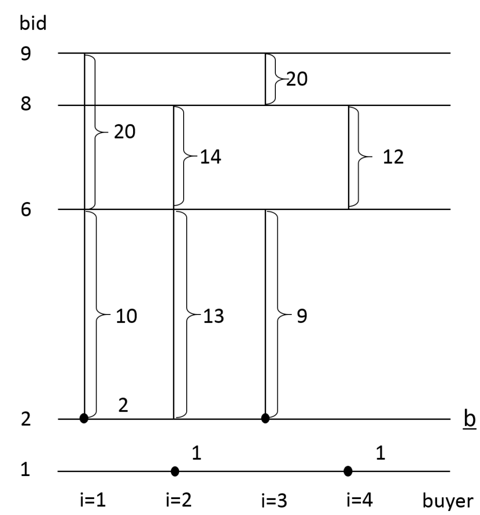


Figure 2: A value's corresponding bid interval is indicated by braces. A dot implies a positive probability (point mass). When Buyer 1 with value 2 and Buyer 3 with value 9 both bid 2. According to Assumption 1, Buyer 3 wins in this case.

3 OVERVIEW OF OUR ALGORITHM

As mentioned in Section 1.2, our algorithm falls in the backward-shooting category. In first-price auctions, if the largest winning bid $\bar{b}$ is given, our theoretical analysis enables us to compute the buyers' bid distributions all the way down to the smallest winning bid $\underline{b}$. However, it turns out that we only know how to compute the smallest winning bid $\underline{b}$ (Theorem 2.2), but have no idea about $\bar{b}$. Therefore, the backward-shooting algorithms just repeatedly guess $\bar{b}$ and see if the computed $\underline{b}(\bar{b})$ matches the actual $\underline{b}$, reducing the computation of the BNE to a searching problem. An overview of the backward-shooting algorithm is shown in Algorithm 1, where a binary search algorithm is used.

Algorithm 1: Overview of the backward-shooting algorithm

Input : Buyers' value distributions G_i.
Output : Buyers' bid distributions F_i.

Compute the smallest winning bid $\underline{b}$ using Lemma 2.2 ;
$UB \leftarrow \max\{\cup_{i \in N} \text{supp}(G_i)\}$, $LB \leftarrow 0$;
while *some exit condition is not met* **do**
  $\bar{b} \leftarrow \frac{1}{2}(UB + LB)$;
  Compute F_i all the way down from $\bar{b}$ to the corresponding smallest winning bid $\underline{b}(\bar{b})$;
  if $\underline{b}(\bar{b}) > \underline{b}$ **then**
    $UB \leftarrow \bar{b}$;
  else
    $LB \leftarrow \bar{b}$;
return F_i;

In Algorithm 1, UB and LB are the upper bound and the lower bound of the largest winning bid. According to Lemma 2.2, the smallest winning bid $\underline{b}$ can be easily determined. The exit condition measures how close our guess of $\bar{b}$ is to the actual largest winning bid, for example, we can check if $UB - LB < \epsilon$ or we can compare whether $\underline{b}(\bar{b})$ is close enough to $\underline{b}$. And guaranteed by Theorem 4.10, 4.12 and Corollary 4.11, we know that $\underline{b}(\bar{b})$ is continuous and increasing with respect to the current guess $\bar{b}$. Therefore the while loop will stop in the end as long as the guess $\bar{b}$ is close enough to the true one. However, the while loop may not end in an acceptable time for some instances shown in Figure 4.

Algorithm 2: Compute $\underline{b}(\bar{b})$ given guess $\bar{b}$.

Input : the largest winning bid guess $\bar{b}$.
Output : the smallest winning bid $\underline{b}(\bar{b})$.

Initialize $b \leftarrow \bar{b}, \Lambda(b) \leftarrow \emptyset$;
Update $\Lambda(b)$ by repeatedly adding buyers to $\Lambda(b)$ according to Theorem 4.6;
while $|\Lambda(b)| \geq 2$ *and* $b > 0$ **do**
  Predict the next change position b' of the bidding set according to Theorem 4.8 and 4.6;
  Set $b \leftarrow b'$;
  Update the bidding set $\Lambda(b')$ by removing buyers according to Theorem 4.8;
  Update the bidding set $\Lambda(b')$ by adding buyers according to Theorem 4.6;
return $\underline{b}(\bar{b}) = b$;

In the continuous value distribution case, the computation of $\underline{b}(\bar{b})$ given $\bar{b}$ is done through solving ordinary differential equations. However, in the discrete case, we compute $\underline{b}(\bar{b})$ with Algorithm 2. We define a core structure called the "bidding set" (Definition 4.1), and starting from $\bar{b}$, we update the bidding set as the bid goes down (i.e., compute where each buyer enters or leaves the bidding set with all his values). Each buyer enters the bidding set with his largest "unconsumed" value when certain conditions are met (Theorem 4.6), and leaves the bidding set when the probability of the corresponding value is used up (Theorem 4.8), hence the value is "consumed". After all buyers consumed all their values, the corresponding $\underline{b}(\bar{b})$ is found. In our setting, Algorithm 2 is used in Algorithm 1 as a sub-routine.

4 THE BIDDING SET

Our objective is to compute every buyer's strategy in BNE. Since a buyer's bidding strategy is monotone, it suffices to compute the bid distribution because we can map a buyer's value to a bid with the same quantile in his bid distribution. However, when the bid support is not continuous, there might exist multiple bids with the same quantile. We introduce a useful tool called the "bidding set", and analyze how the bidding set changes in the bid space. With the analysis of the structure, an algorithm of computing the BNE can be naturally derived.

Definition 4.1 (Bidding set and waiting list). The set of buyers whose bidding strategies include bid x is called the *bidding set*, denoted by $\Lambda(x)$, i.e., $\Lambda(x) = \{i \mid x \in S_i\}$. The set of other buyers $N - \Lambda(x)$ is called the *waiting list* at bid x.

When there is no ambiguity, we use Λ instead. The following theorem is about the relationship between the bid distribution and Λ. For any buyer set Λ, let $F_\Lambda(x)$ denote the product of the cumulative bid distribution of buyers in Λ, i.e., $F_\Lambda(x) = \prod_{i \in \Lambda(x)} F_i(x)$. We abuse notation and use $v_i(x)$ to represent player i's value when he bids x in the equilibrium ($v_i(x)$ is well-defined according to Lemma 2.5).

Before discussing how the bidding set Λ changes in the bid space, let's first consider some properties of the bidding set. Define function

$$h_i(x) = \frac{1}{|\Lambda(x)| - 1}\left(\sum_{j \in \Lambda(x)} \frac{1}{v_j(x) - x}\right) - \frac{1}{v_i(x) - x}, \forall i \in \Lambda(x).$$

Theorem 4.2. *Suppose $\Lambda(x)$ does not change in bid interval (b_1, b_2), and $v_i(x)$ is constant for $x \in (b_1, b_2)$, $i \in \Lambda(x)$. Then the bid distribution of every buyer in Λ is differentiable in this interval. In fact, for any $x \in (b_1, b_2)$ we have*

$$\frac{f_i(x)}{F_i(x)} = h_i(x), \forall i \in \Lambda.$$

All proofs are omitted due to the lack of space.

If we know what the bidding set $\Lambda(x)$ is for every possible x in the BNE, we can construct each buyer's bid distribution $f_i(x)$ according to Theorem 4.2. Therefore, the rest of this section is devoted to the analysis of how the bidding set changes.

Now we discuss the basic components $S_i(v_i^j)$. In the continuous case, it is known that the support of the bidding strategy is connected for every buyer [14]. This result no longer holds in the discrete value setting. In Example 2.6, Buyer 3's possible bids have two connected parts. However, we can prove a weaker version of this structural result.

Theorem 4.3. *$S_i(v_i^j)$ is an interval when $v_i^j \geq \underline{b}$.*

4.1 Change Positions of the Bidding Set

In this section, we consider some properties of the bidding set when it changes. These results are helpful for computing these change positions.

Definition 4.4. When bidding set changes at x, we use $\Lambda^+(x)$ and $\Lambda^-(x)$ to denote the buyers who bid in the upper neighborhood and lower neighborhood around x, i.e.,

$$\Lambda^+(x) = \{i \mid \exists \epsilon > 0, (x, x+\epsilon) \subseteq S_i\},$$
$$\Lambda^-(x) = \{i \mid \exists \epsilon > 0, (x-\epsilon, x) \subseteq S_i\}.$$

Clearly, when a bidding interval $S_i(v_i^j)$ starts or ends at a certain bid x, $\Lambda(x)$ changes. Therefore, to characterize how the bidding set changes, it suffices to determine when a bid interval $S_i(v_i^j)$ starts and ends, or equivalently, when a buyer enters and leaves the bidding set. Our method falls in the backward-shooting category, thus we compute the buyers' bidding strategy from the largest winning bid all the way down.

Definition 4.5. We say a buyer *enters* the bidding set with value v_i at bid x if $x = \max S_i(v_i)$. Similarly, we say a buyer *leaves* the bidding set if $x = \min S_i(v_i)$.

Remark 1. *Notice that entering the bidding set with value v_i is different from:*

$$i \notin \Lambda^+(x), \text{ and } i \in \Lambda^-(x).$$

The reason is that it is possible for the buyer to leave the bidding set with another value v_i' and enters immediately at the same point, but with a different value v_i.

4.2 When to Enter the Bidding Set

The following theorem determines when should a buyer enter the bidding set.

Theorem 4.6. *Suppose buyer i has the largest unconsumed value v_i in the waiting list, he will enter the bidding set when one of the following conditions is satisfied.*

- *$|\Lambda| \leq 1$ and $v_i > x$;*
- *$\frac{1}{v_i - x} \leq \frac{1}{|\Lambda|-1} \sum_{i \in \Lambda} \frac{1}{v_j(x) - x}$ (or equivalently $h_i(x) \geq 0$) and $v_i > x$.*

We provide an example to show how to apply Theorem 4.6.

Example 4.7. Consider bid 6 in Example 2.6 and Figure 2. Bidding set $\Lambda^+(6)$ is $\{1, 2, 4\}$, with corresponding values 20, 14, and 12. all buyers in $\Lambda^+(6)$ have consumed the probability of their current value at bid 6, and they all leave the bidding set. Thus the bidding set becomes empty and the waiting list contains all buyers. Buyer 2 and Buyer 1 enters the bidding set sequentially according to the first condition in Theorem 4.6. Buyer 3 enters the bidding set according to the second condition $1/(9-6) \leq 1/(10-6) + 1(13-6)$. However, Buyer 4 does not enter since his value is smaller than the current bid. After the update, we have $\Lambda^-(6) = \{1, 2, 3\}$.

4.3 When to Exit the Bidding Set

The probability of a buyer's value being v_i should equal the probability that he bids in the interval $S_i(v_i)$. By Theorem 4.3, the bid set $S_i(v_i)$ of a specific value v_i is a connected interval. Therefore, once we know the maximum bid in $S_i(v_i)$, buyer i will not leave the bidding set until value v_i is consumed.

Theorem 4.8. *Buyer i with value v_i^k leaves the bidding set at x when the cumulative probability of bidding set equals to the probability of the value, i.e., $F_i\left(\max S_i(v_i^k)\right) - F_i(x) = G_i(v_i^k) - G_i(v_i^{k-1})$.*

Example 4.9. Consider Example 2.6 and Figure 2, $S_1(20)$ begins at bid 9 and consumes the probability of value 20 at bid 6. The probability that buyer 1 bids in $S_1(20)$ is

$$F_1(9) - F_1(6) = \frac{11}{20-9} - \frac{77}{48}\sqrt{\frac{10-6}{(9-6)(13-6)}} = 1 - \frac{11}{24}\sqrt{\frac{7}{3}},$$

which equals the probability of value 20.

4.4 Monotonicity of Change Points

Now we present some monotonicity results in the discrete setting. These results are similar to the continuous case, but with different proofs. Define

$$p_i^j = \ln G_i(v_i^j) - \ln G_i(v_i^{j-1}), \forall i = 1, \ldots, n, j = 2, \ldots, i_k.$$

So $\{p_i^j\}_{i=1,\ldots,n,j=2,\ldots,i_k}$ uniquely determines the value distribution G. When there is no ambiguity, we use $\{p_i^j\}$ for simplicity. We use $\mathcal{E}(\bar{b}, \{p_i^j\})$ to denote the set of bidding intervals given by Algorithm 2 with a guessed largest bid $\bar{b}$ and distribution G.

Theorem 4.10. *The extreme points of every bid interval in $\mathcal{E}(\bar{b}, \{p_i^j\})$ is monotone in $\bar{b}$.*

The proof is different from the continuous distribution. We prove it by analyzing the algorithm directly.

Corollary 4.11. *The position $\underline{b}(\bar{b})$ where Algorithm 2 stops is strictly monotone in $\bar{b}$.*

It is possible that some bid intervals remain at the same positions. But the position where the algorithm stops increases strictly. Next we prove the continuity of the endpoints of each bid interval.

Theorem 4.12. *The limit of each bid interval constructed by Algorithm 2 with the largest winning bid approaching to $\bar{b}^1$, is same as the bid interval constructed with the largest winning bid $\bar{b}^1$.*

5 EXISTENCE AND UNIQUENESS OF THE BNE

5.1 Existence

In Algorithm 2, if the point $\underline{b}(\bar{b})$ where the algorithm terminates does not match the actual smallest winning bid $\underline{b}$, the bidding strategies we get do not form a BNE. But we show that if it does match $\underline{b}$, then the corresponding strategies do form a BNE.

Lemma 5.1. *If $\underline{b}(\bar{b})$ matches the smallest winning bid $\underline{b}$, the bidding strategies given by Algorithm 2 is indeed a BNE.*

Theorem 5.2. *A Bayesian Nash Equilibrium always exists when buyers have discrete value distributions.*

Corollary 5.3. *Buyers with identical value distributions have identical bidding strategies in the BNE. Furthermore, if all buyers have identical value distributions, i.e. symmetric distributions, the Bayesian Nash equilibrium is also symmetric.*

5.2 Uniqueness

According to the monotonicity of $\underline{b}(\bar{b})$, we can check whether the guessed bid is too high or too low. Corollary 4.11 implies that only one guess of the largest winning bid $\bar{b}$ can possibly equate the corresponding $\underline{b}(\bar{b})$ and the actual smallest winning bid $\underline{b}$.

Theorem 5.4. *There exists a unique Bayesian Nash equilibrium when buyers have discrete value distributions.*

Remark 2. *By uniqueness, we mean the equilibrium above $\underline{b}$ is unique. In Example 2.6, we can change buyer 2's bids below 2 to any other bids below 2, and still get an equilibrium. But we only focus on the structure above the smallest winning bid.*

In the continuous distribution case, in order to prove the uniqueness result, Lebrun [15] relies on the assumption that the distribution is strictly log-concave (f_i/F_i is strictly decreasing) at the highest lower extremity of the supports, i.e., v_1^1 in our example. Briefly speaking, he uses this assumption to handle the case where some buyers always give bids that is larger than $\underline{b}$. However, in the discrete setting, according to our results, we do not need to deal with such cases.

6 EXPERIMENTS

Algorithm 1 computes the BNE of the first-price auction by repeatedly guessing the largest winning bid $\bar{b}$. According to Theorem 4.2, the change points of the bidding set completely determines the buyers' bidding strategies in the BNE. And given a guess $\bar{b}$, we compute change points of the bidding set $\Lambda(x)$ using Algorithm 2.

6.1 Accuracy Comparison between Continuous Algorithms and Our Algorithm

When buyers have discrete value distributions, one natural way of computing the BNE is to approximate the discrete value distribution with a continuous one. Of course, there are infinitely many ways of approximation. Our choice is to replace a discrete value with a "triangle" probability density function centered at that value, and cover the interval $[v_i^1, v_i^{d_i}]$ with a small uniform distribution.

For simplicity, we assume that for all i and d, $|v_i^{d+1} - v_i^d| > 2w$ and $w < v_i^d < 1 - w$. Note that although we stick to this approximation throughout this section, our analysis also applies to other possible ways of approximation.

We implemented the continuous backward-shooting algorithm using the characterization by Maskin and Riley [18]:

$$\frac{\mathrm{d}t_i(b)}{\mathrm{d}b} = \frac{G_i(t_i(b))}{g_i(t_i(b)}\left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{t_j(b)-b}\right)-\frac{1}{t_i(b)-b}\right], \forall i \in [n]. \quad (1)$$

Using a smaller step size s could significantly increase the number of loops inside Algorithm 3. However, using a smaller s could also make the computation result more accurate. Also, when $t_i \notin \cup_{i,d}[v_i^d - w, v_i^d + w]$, i.e., $f_i(t_i) = \epsilon$, the ratio $\frac{F_i(t_i)}{f_i(t_i)}$ could be very large. Therefore, using a relatively large s could lead to a sudden decrease of t_i, causing the program to skip other v_i^d's in between.

Algorithm 3: The continuous backward-shooting algorithm

Input: step s, max winning bid guess $\bar{b}$
Output: the min winning bid

```
b ← b̄, t_i ← 1;
while t_i > b, ∀i ∈ [n] do
    for i ∈ [n] do
        compute t'_i(b) according to Equation (1);
        t_i ← t_i − t'_i(b) · s;
    b ← b − s;
return b;
```

Example 6.1. Consider the case where there are 6 buyers. Their value distributions are as follows:

$$(v_1^1, v_1^2, v_1^3) = (0.08, 0.2, 0.8), (v_2^1, v_2^2, v_2^3) = (0.09, 0.3, 0.9),$$
$$(v_3^1, v_3^2, v_3^3) = (0.07, 0.12, 0.7), (v_4^1, v_4^2, v_4^3) = (0.07, 0.12, 0.7),$$
$$(v_5^1, v_5^2, v_5^3) = (0.07, 0.12, 0.7), (v_6^1, v_6^2, v_6^3) = (0.04, 0.12, 0.8),$$
$$(P_1^1, P_1^2, P_1^3) = (0.2, 0.76, 0.04), (P_2^1, P_2^2, P_2^3) = (0.3, 0.36, 0.34),$$
$$(P_3^1, P_3^2, P_3^3) = (0.3, 0.36, 0.34), (P_4^1, P_4^2, P_4^3) = (0.3, 0.36, 0.34),$$
$$(P_5^1, P_5^2, P_5^3) = (0.2, 0.15, 0.65), (P_6^1, P_6^2, P_6^3) = (0.2, 0.15, 0.65),$$

where P_i^d is the probability of buyer i's value being v_i^d One can avoid the above problem by carefully tuning the parameter s. However, the possible large values of $\frac{F_i(t_i)}{f_i(t_i)}$ can cause other problems that may not have easy solutions:

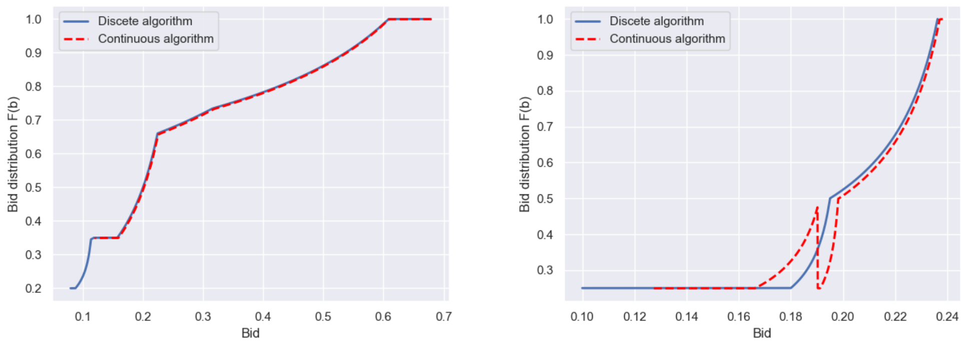
(1) a large $\frac{F_i(t_i)}{f_i(t_i)}$ leads to a large $t_i'(b)$, meaning that $t(b)$ decreases much faster than b. This may cause the algorithm to terminate early if $t(b)$ becomes smaller than b.
(2) in the next loop, $t_i'(b)$ can be negative and t_i will oscillate as a result (see Figure 3b).

To understand the first problem, consider the following example: Figure 3a shows the bidding strategy of Buyer 5 in the above example. The minimum winning bid computed by the continuous algorithm is about 0.12, while the actual minimum winning bid is 0.08, indicating that Algorithm 3 terminates early. The reason is that during the execution of Algorithm 3, when b is near 0.12, t_2 is near 0.25, but t_2' is over 1500. This means that a slight decrease in b could lead to a significant drop in t_2, making $t_2 < b$ and terminating the algorithm.

To understand the second problem, consider the case where t_i is close to b in the BNE for some b. Then t_i' is very likely to become negative according to Equation (1), and it is not clear how we could avoid such problems since the computation of t_i' is independent of s.

Example 6.2. Consider the case where there are 3 buyers. Their value distributions are as follows:

$$(v_1^1, v_1^2, v_1^3) = (0.1, 0.2, 0.25), (v_2^1, v_2^2, v_2^3) = (0.1, 0.2, 0.25),$$



(a) Early termination of continuous algorithm **(b) The oscillation of continuous algorithms.**

Figure 3: Accuracy comparison. The two curves in right figure correspond to buyer 1's equilibrium bidding strategy computed by both algorithms.

$$(v_3^1, v_3^2, v_3^3) = (0.1, 0.2, 0.25), (P_1^1, P_1^2, P_1^3) = (0.25, 0.25, 0.5),$$
$$(P_2^1, P_2^2, P_2^3) = (0.05, 0.45, 0.5), (P_3^1, P_3^2, P_3^3) = (0.05, 0.45, 0.5).$$

Figure 3b shows the bidding strategy of Buyer 1 in BNE. The bid distribution $F(b)$ is computed according to the corresponding $t(b)$. The oscillation in the curve indicates the oscillation in $t(b)$. As shown in the figure, the oscillation occurs when $v(b)$ is close to b.

Although the continuous backward-shooting algorithm can be problematic, our experiments show that it has good performance if we only need to figure out the maximum winning bid $\bar{b}$. The problems we mentioned mainly affects the computation of the strategies for smaller values. The computation of this part of strategies also suffers from sensitivity issues, as discussed in [8]. To overcome this difficulty, Fibich and Gavish [8] proposed another algorithm where guessing the maximum winning bid is no longer needed. We also try their algorithms and conduct experiments. When it comes to approximating discrete value distributions with continuous ones, the computation still suffers from sensitivity issues due to very high condition numbers, implying that the sensitivity might be an intrinsic issue of this problem.

However, our discrete algorithm does not use continuous distributions to approximate discrete ones, thus can avoid all the above problems. The computational complexity only depends on the number of value points. Also, our algorithm does not have the oscillation problem or sensitivity issues, since our theoretical analysis already characterizes the structure of the solution, and involves none of the sensitivity computation mentioned above. Therefore, our discrete algorithm can provide a much accurate solution compared to other ones.

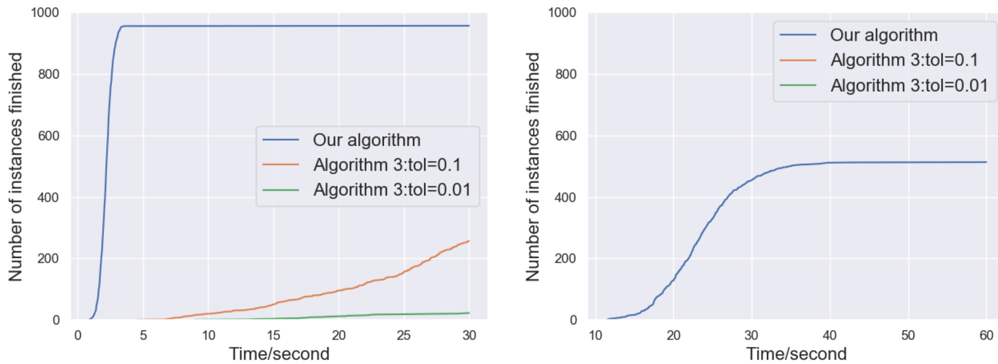
6.2 Running Time Comparison between Continuous Algorithms and Our Algorithm

In this section, we compare the running time of previous continuous algorithms and our discrete algorithm. Since the algorithm provided by Fibich and Gavish [8] often gives a condition number issue, we only compare our algorithm with Algorithm 3 in these experiments. We conduct experiments for three different settings. For each setting, the experiment setup is as follows: We generate 1000 first price auction instances, with each containing n buyers. For each buyer, we sample d different values from the interval $[0, 1]$, and the corresponding value distribution is also randomly generated for each buyer. Then both our algorithm and Algorithm 3 are applied to compute the Bayes-Nash equilibrium. Both algorithms need to guess the maximum winning bid, so the final computed minimum winning bid would be different from the actual minimum winning bid. Therefore we also set a tolerance parameter tol, which serves as a stopping criterion (i.e., the algorithm terminates when the difference between the computed minimum winning bid and the actual minimum winning bid is smaller than tol). As these algorithm runs, we record the running time of the algorithms on each instance. Considering that in some cases, the algorithms may take a very long time to terminate, we set another deadline parameter T, and kill the process once the running time exceeds T. During the experiments, we make sure that no other programs are running and at any time, only one algorithm is running on one instance. Also, we only compare the running time in these experiments, so detailed solution qualities are ignored.

The parameters for the three settings are as follows: 1) small: $n = 5, d = 5, T = 30$ seconds, 2) medium: $n = 10, d = 10, T = 60$ seconds; 3) large:$n = 100, d = 100, T = 60$ seconds.

For all the settings, we run our algorithm with $tol = 10^{-8}$, and run Algorithm 3 twice with $tol = 0.1$ and $tol = 0.01$. The experiment results are shown in Figure 4.

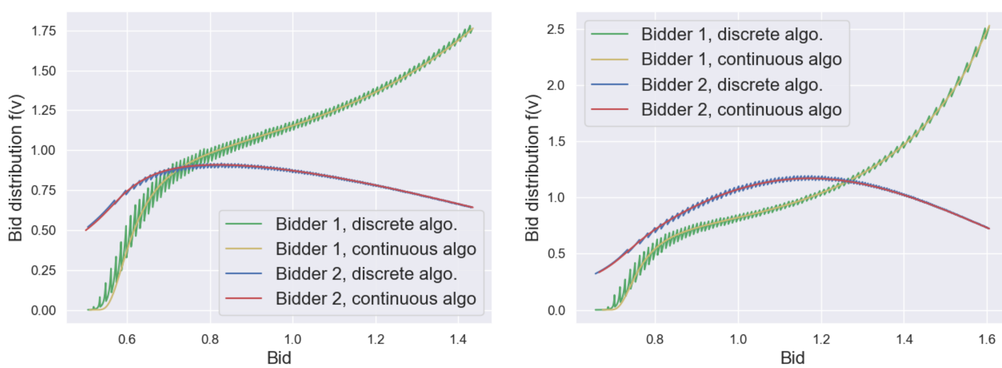
For the 1000 small instances, although we set a much smaller tolerance value $tol = 10^{-8}$ for our algorithm, our algorithm finishes on almost all instances (955) within the 30 seconds deadline, Algorithm 3 finishes on only 256 instances when $tol = 0.1$ and on only 22 instances when $tol = 0.01$. For medium instances, our algorithm finishes on 512 of them, while Algorithm 3 does not finish on any instance within the deadline. And for large instances, no algorithm ever finishes on any instance within the deadline. It is interesting that our algorithm either finishes very quickly, or does not finish after a relatively long time. For example, among the finished 955



(a) Small instances

(b) Medium instances

Figure 4: Running time comparison. The y-axis is the cumulative distribution of the running time for the 1000 instances (i.e., the number of instances finished within the corresponding time period). For large instances, no algorithm can finish the computation within the 60 seconds deadline, thus are not shown.



(a) Case 1

(b) Case 2

Figure 5: Comparison between the continuous algorithm and our discrete algorithm.

small instances, almost all of them finish within the first 3.5 seconds. This is also true for medium instances. The reason behind this observation is still unknown. We believe this is closely related to specific value distributions in the instances.

We also compare the performance of the continuous algorithm and our discrete algorithm when the value distribution is continuous (see Figure 5). In our experiments, there are two bidders with different value distributions. We consider two cases. In the first case, bidder 1's value follows uniform distribution over the interval $[1, 3]$, and bidder 2's value follows uniform distribution over interval $[0, 2]$. In the second case, bidder 1's value distribution is the same while bidder 2's value follows the distribution below:

$$G_2(v) = 0.25v^2, \forall v \in [0, 2].$$

To run the discrete algorithm, we first need to convert the continuous distributions to discrete ones. For each bidder, we split the support of his value distribution to 120 non-overlapping small intervals of the same length. For each small interval, we use the center of it as the representative, and set its probability to be that of the bidder's value lying in the interval.

The results of the experiments are shown in Figure 5. Both algorithms give almost the same maximum winning bid and the minimum winning bid. The probability distribution functions provided by the two algorithms are also quite similar, meaning that the discrete algorithm can approximate the solution well with a fine discretization scheme.

It is interesting that as the bid gets smaller, the "sawtooth" phenomenon becomes more obvious. This is due to the fact that when the bid is small, $v_i(b)$ becomes close to b, and $f_i(b)$ is more sensitive to the change of b as it depends on $\frac{1}{v_i(b)-b}$.

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